Midterm
Math 309: Introduction to knot theory
February 14, 2019

Instructions: This exam has **3 questions**. The first is a series of multiple choice problems worth a total of 10 points (you do not need to show your work). The second and third are multi-part questions (the point value of each sub-question has been indicated). Please ensure that your test has 5 pages, including this cover page. With the exception of multiple choice questions, **show all your work**; exam booklets have been provided. **You have 80 minutes to complete the test.**

Important: This test is closed book. There should be nothing on your desk other than this test and something to write with. This includes your cell phones, which you have already turned off.

Problem 1, multiple choice [Total points: 10]
For each question below, choose the best answer in each case. You do not need to justify your answer to receive full credit.

(i) [2 points] The bridge number of a knot is:

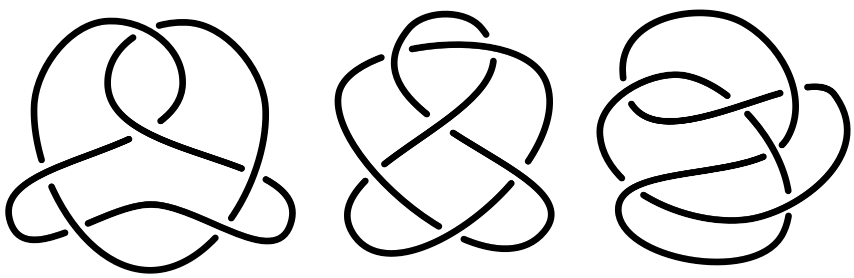
(a) Half the number of critical points in $\mathbb{R}^3$ along the knot viewed as a closed curve in $\mathbb{R}^3$, minimized over all closed curves representing equivalent knots.

(b) The number of maximal overpasses in a diagram, minimized over all diagrams representing the same knot.

(c) Both of the above: definitions (a) and (b) are equivalent.

(d) None of the above.

(ii) [2 points] Consider the tricolourability of following three knots:



[I] **[II]** **[III]**

(a) All three are tricolourable

(b) None of these knots is tricolourable

(c) **[I]** is tricolourable but **[II]** and **[III]** are not.

(d) **[I]** is not tricolourable but **[II]** and **[III]** are.

(iii) [2 points] Consider the knot described by the parametrization

$$\begin{aligned} x(t) &= \big(2+\cos(2t)\big)\cos(5t) \\ y(t) &= \big(2+\cos(2t)\big)\sin(5t) \\ z(t) &= \sin(2t) \end{aligned}$$

This knot has minimal crossing number:

(a) 5

(b) 6

(c) 10

(d) None of the above.

(iv) [2 points] The figure eight knot has Jones polynomial:

(a) Equal to 1.

(b) With non-zero coefficients only on positive powers of t.

(c) That is symmetric, in the sense that the coefficient on t^n is equal to that of t^{-n} for every n.

(d) None of the above.

(v) [2 points] For any knot K, denote the crossing number by $c(K)$ and the unknotting number by $u(K)$.

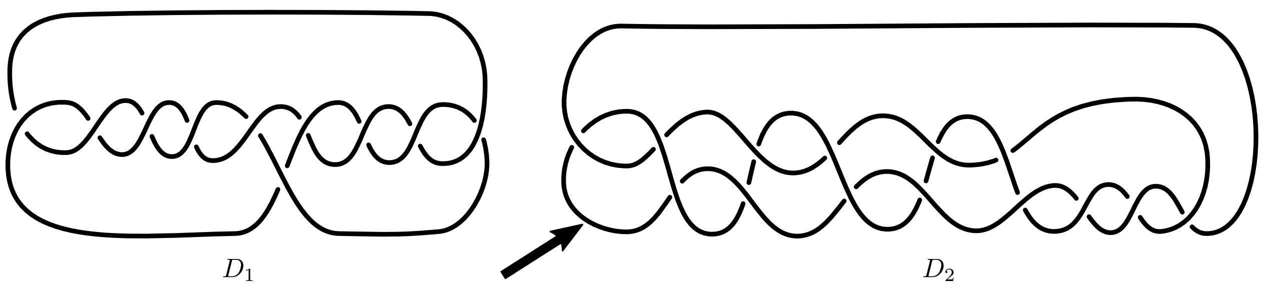
(a) $c(K) \leq u(K)$

(b) $c(K) = u(K)$

(c) $c(K) \geq u(K)$

(d) None of the above.

Problem 2 [Total points: 20 (broken down below)]
Consider the following knot diagrams:



(a) [**4 points**] Starting with diagram D_2, show that these diagrams are equivalent. Show your steps; you need *not* show every Reidemeiser move required. (**Hint:** start at the understrand indicated by the arrow)

(b) [**4 points**] Each of these diagrams describes a rational knot. Find a diagram for the associated rational tangle in each case, using the conventions in Adams. (**Hint:** remember that there are two slightly different procedures, depending on the parity of the number of twist-sets)

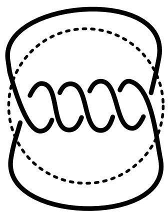
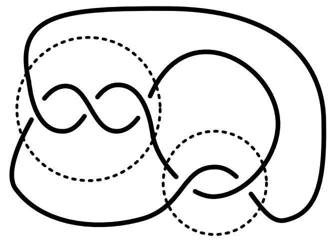
(c) [**2 points**] For each of the rational tangles you found in part **(b)**, find the associated continued fraction expressed in the notation $[a_n \ a_{n-1} \ \cdots \ a_1]$ for integers a_i.

(d) [**2 points**] Verify that the continued fractions you found in part **(c)** agree.

(e) [**4 points**] We have now established, in two different ways, that the diagrams D_1 and D_2 describe the same knot K. Prove that $u(K) \leq 2$, where $u(K)$ denotes the unknotting number of K.

(f) [**4 points**] A recent result of McCoy states that if a knot is alternating and has unknotting number 1 then any alternating diagram for the knot contains an unknotting crossing. Appealing to this result, prove that $u(K) \geq 2$.

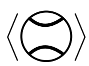
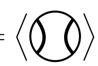
Problem 3 [Total points: 20 (broken down below)]
This problem involves the following knots:

$K_1 =$  $K_2 =$ 

Ultimately, in this problem you will calculate the Jones polynomial for each of these knots; the dashed circles propose tangle decompositions for these knots. These calculations are broken into steps below. Before you start, it will be important to recall that, for any given tangle T, it is always possible to express

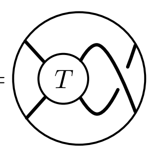
$$\langle T \rangle = P_0 \mathbf{e}_0 + P_\infty \mathbf{e}_\infty$$

where $P_0, P_\infty \in \mathbb{Z}[A, A^{-1}]$ are the coefficient polynomials and

$\mathbf{e}_0 =$  $\mathbf{e}_\infty =$ 

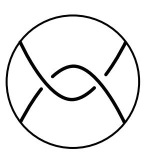
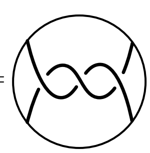
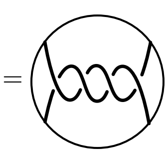
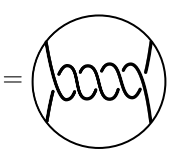
(a) [2 points] Find the polynomials P_0 and P_∞ associated with the tangle 

(b) [4 points] We have seen that, given any tangle T, it is possible to produce a new (different!) tangle by adding a single twist as in the following tangle diagram:

$T^{\text{twist}} =$ 

Compute the coefficient polynomials P_0^{twist} and P_∞^{twist} in terms of A,A^{-1}, P_0, and P_∞. (**Hint:** Resolve the new crossing and use $\langle T \rangle = P_0 \mathbf{e}_0 + P_\infty \mathbf{e}_\infty$)

(c) [4 points] Appealing to the formula you found in part **(b)**, calculate the bracket of the following tangles:

$T_2 =$  $T_3 =$  $T_4 =$  $T_5 =$ 

Show all your work, and collect your final answer in a single table. (**Hint:** Start with $\langle T_2 \rangle$ since $\langle T_i \rangle$ will depend on $\langle T_{i-1} \rangle$ in general)

(d) [2 points] Appealing to your work in part **(c)**, compute $\langle K_1 \rangle$.

(e) [4 points] Similarly, compute $\langle K_2 \rangle$.

(f) [4 points] Finally, conclude that $K_1 \nsim K_2$ by computing $V_{K_1}(t)$ and $V_{K_2}(t)$ and showing that these are not equal as polynomials in $\mathbb{Z}[t, t^{-1}]$.