Problem 1, multiple choice [Total points: 10]
For each question below, choose the best answer in each case. You do not need to justify your answer to receive full credit.

(i) [2 points] The bridge number of a knot is:

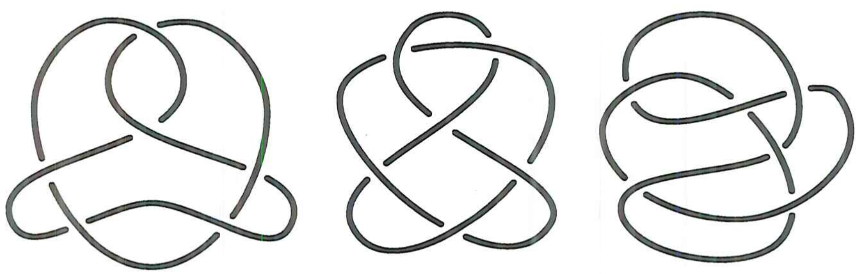
(a) Half the number of critical points in $\mathbb{R}^3$ along the knot viewed as a closed curve in $\mathbb{R}^3$, minimized over all closed curves representing equivalent knots.

(b) The number of maximal overpasses in a diagram, minimized over all diagrams representing the same knot.

(c) Both of the above: definitions (a) and (b) are equivalent.

(d) None of the above.

(ii) [2 points] Consider the tricolourability of following three knots:



[I] [II] [III]

(a) All three are tricolourable

(b) None of these knots is tricolourable

(c) [I] is tricolourable but [II] and [III] are not.

(d) [I] is not tricolourable but [II] and [III] are.

(iii) [2 points] Consider the knot described by the parametrization

$$x(t) = \big(2 + \cos(2t)\big)\cos(5t)$$
$$y(t) = \big(2 + \cos(2t)\big)\sin(5t)$$
$$z(t) = \sin(2t)$$

This knot has minimal crossing number:

(a) 5

(b) 6

(c) 10

(d) None of the above.

(iv) **[2 points]** The figure eight knot has Jones polynomial:

(a) Equal to 1.

(b) With non-zero coefficients only on positive powers of t.

(c) That is symmetric, in the sense that the coefficient on t^n is equal to that of t^{-n} for every n.

(d) None of the above.

(v) **[2 points]** For any knot K, denote the crossing number by $c(K)$ and the unknotting number by $u(K)$.

(a) $c(K) \leq u(K)$

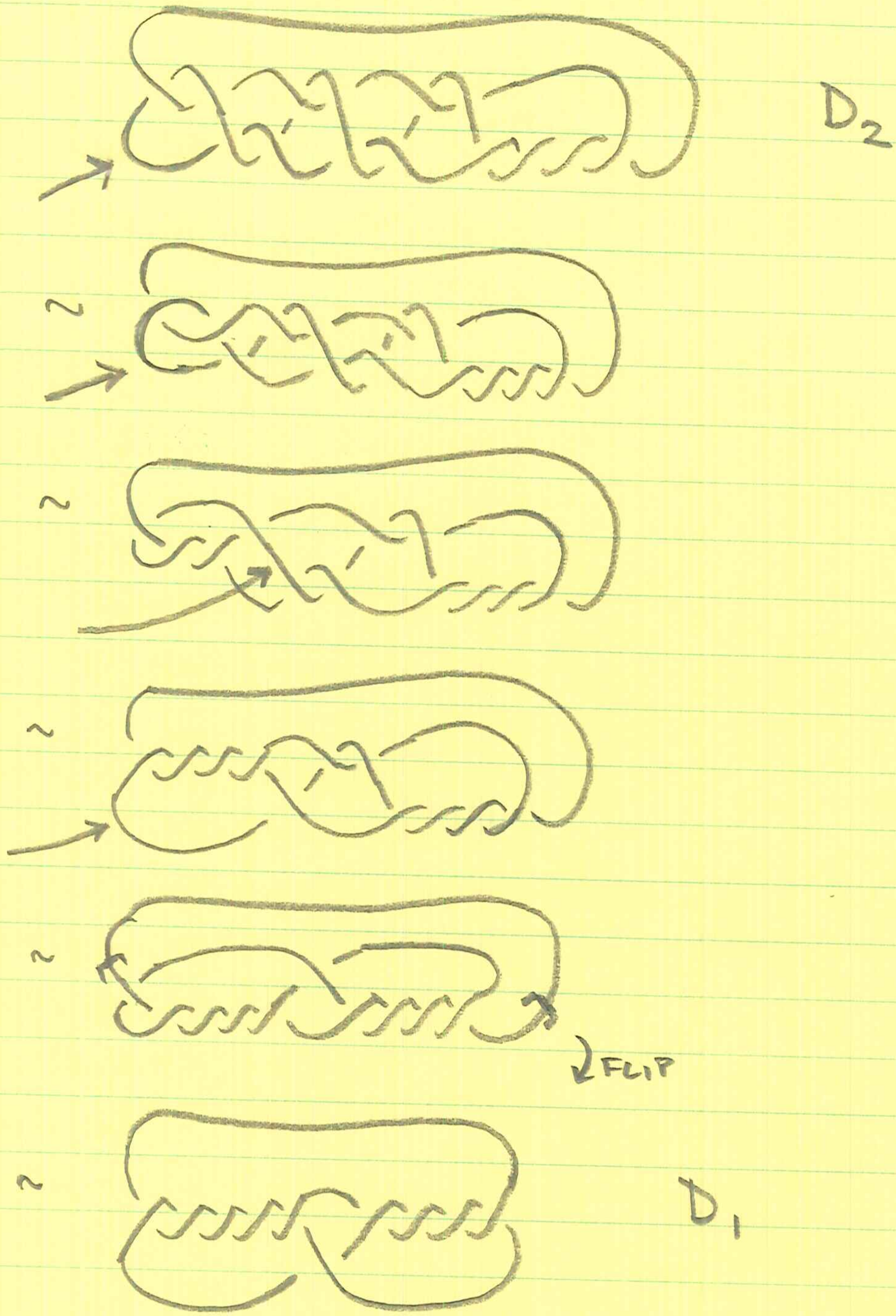
(b) $c(K) = u(K)$

(c) $c(K) \geq u(K)$

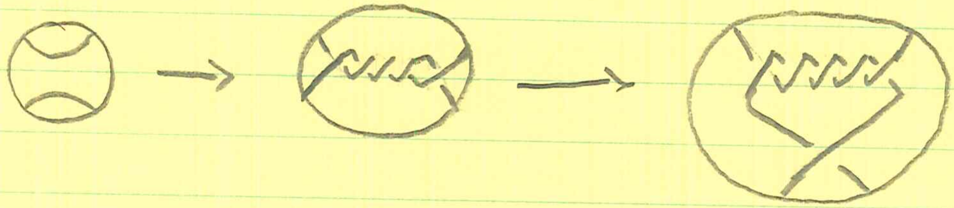
(d) None of the above.

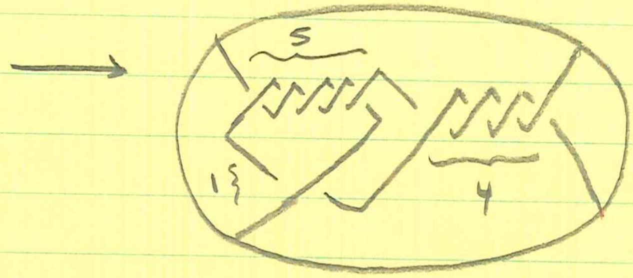
PROBLEM 2

(a)



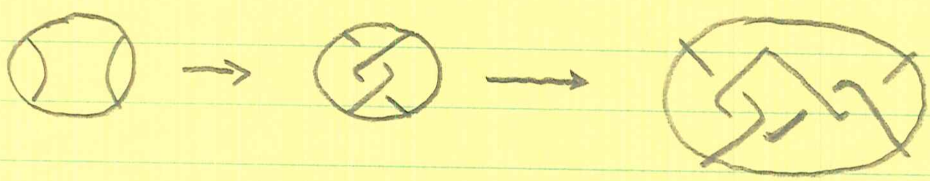
(b/c) D_1 HAS AN ODD NUMBER OF TWIST SITES:

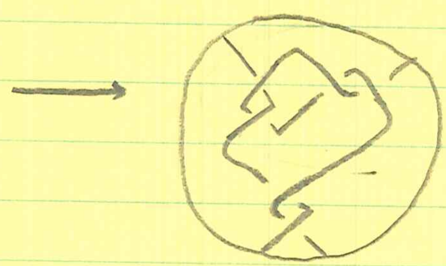




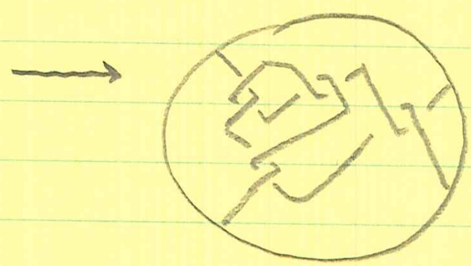
[5 1 4]

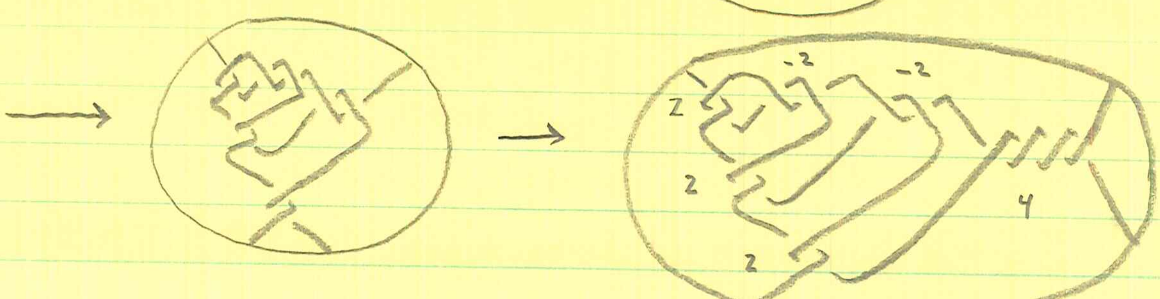
D_2 HAS AN EVEN NUMBER OF TWIST SITES:





[2 -2 2 -2 2 4]





(d) $[5\ 1\ 4] = 4 + \cfrac{1}{1 + \cfrac{1}{5}} = 4 + \dfrac{5}{6} = \dfrac{29}{6}$

$$[2\ {-2}\ 2\ {-2}\ 2\ 4] = 4 + \cfrac{1}{2 + \cfrac{1}{-2 + \cfrac{1}{2 + \cfrac{1}{-2 + \cfrac{1}{2}}}}}$$

$$= 4 + \cfrac{1}{2 + \cfrac{1}{-2 + \cfrac{1}{2 - \cfrac{2}{3}}}}$$

$$= 4 + \cfrac{1}{2 + \cfrac{1}{-2 + \cfrac{3}{4}}}$$

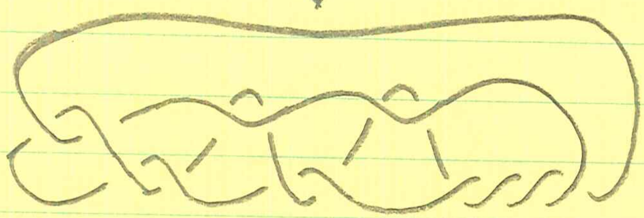
$$= 4 + \cfrac{1}{2 - \cfrac{4}{5}}$$

$$= 4 + \frac{5}{6}$$

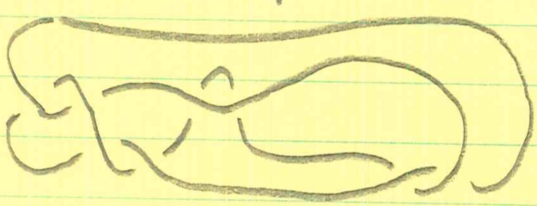
$$= \frac{29}{6}$$

(e)

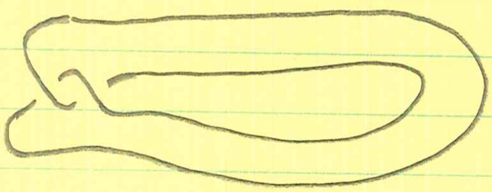
$\downarrow u$



$\downarrow$ R2 $\times$ 3



$\downarrow$ R2 $\times$ 3

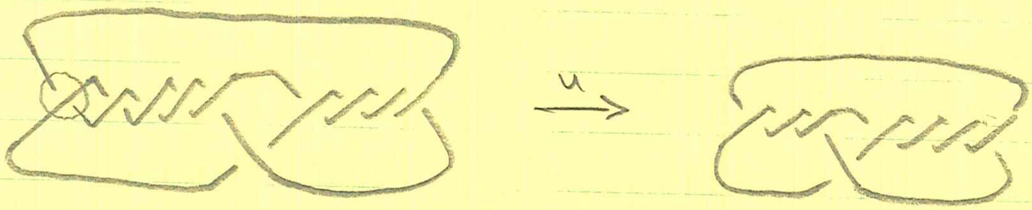


$\downarrow$ R1 $\times$ 2

THIS UNKNOTS D_2 WITH <u>TWO</u> CROSSING CHANGES HENCE

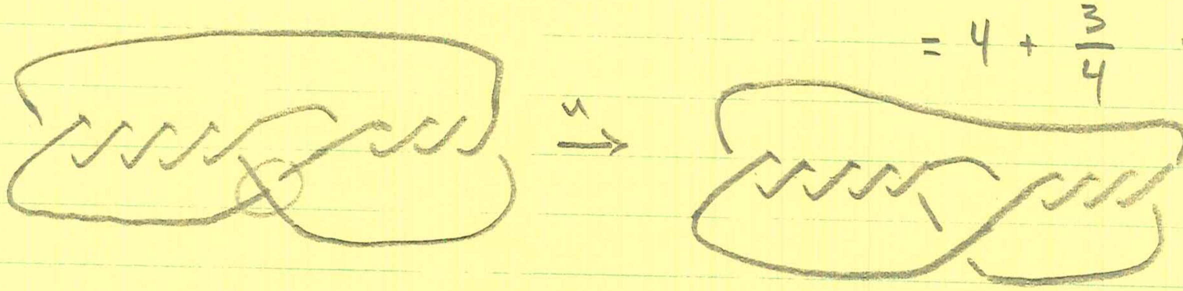
$$u(K) \leq 2$$

(f) SINCE D_1 IS AN ALTERNATING DIAGRAM, IT IS ENOUGH TO SHOW THAT IT IS <u>NOT</u> POSSIBLE TO UNKNOT D_1 WITH ONLY <u>ONE</u> CROSSING CHANGE.

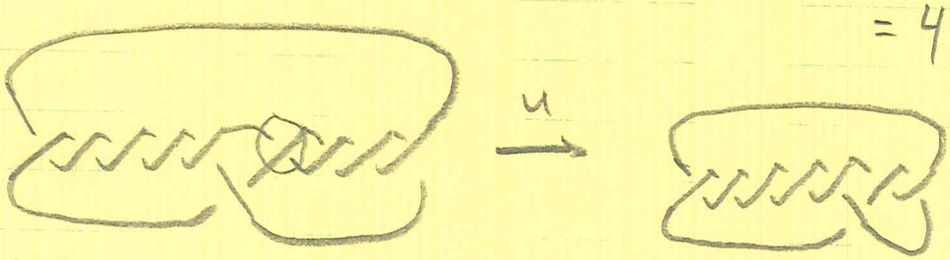


(SAME FOR ANY OF THESE FIVE CROSSINGS)

$$[3\ 1\ 4] = 4 + \frac{1}{1+\frac{1}{3}} = 4 + \frac{3}{4} \neq 0, \frac{1}{N}$$



$$[5\ {-1}\ 4] = 4 + \frac{1}{-1+\frac{1}{5}} = 4 - \frac{5}{4} \neq 0, \frac{1}{N}$$

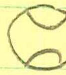


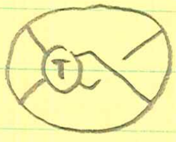
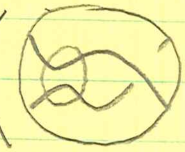
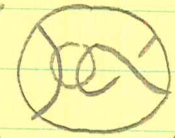
(SAME FOR ANY OF THESE CROSSINGS)

$$[5\ 1\ 2] = 2 + \frac{1}{1+\frac{1}{5}} = 2 + \frac{5}{6} \neq 0, \frac{1}{N}$$

THEREFORE $u(K) > 1$ (OR $u(K) \geq 2$).

PROBLEM 3

(a) $\langle$  $\rangle = A \langle$  $\rangle + A^{-1} \langle$  $\rangle$ so $P_0 = A$, $P_\infty = A^{-1}$

(b) $\Big\langle$  $\Big\rangle = P_0 \Big\langle$  $\Big\rangle + P_\infty \Big\langle$  $\Big\rangle$

$= AP_0 \langle$  $\rangle + A^{-1}P_0 \langle$  $\rangle$

$+ AP_\infty \langle$  $\rangle + A^{-1}P_\infty \langle$  $\rangle$

$= AP_0 \langle$  $\rangle + \left(A^{-1}P_0 + AP_\infty + A^{-1}P_\infty \gamma \right) \langle$  $\rangle$

WHERE $\gamma = -A^{-2} - A^2$. SO

$$P_0^{\text{TWIST}} = AP_0$$

$$P_\infty^{\text{TWIST}} = A^{-1}P_0 + \left(A + A^{-1}(-A^{-2} - A^2)\right) P_\infty$$

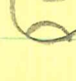
$$= A^{-1}P_0 - A^{-3} P_\infty$$

(c) LET $\langle T_2 \rangle = P_0^{(2)} \langle$  $\rangle + P_\infty^{(2)} \langle$  $\rangle$

THEN $P_0^{(2)} = \left(P_0^{(1)}\right)^{\text{TWIST}} = AA = A^2$

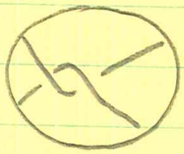
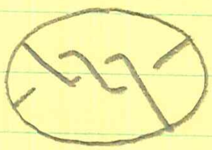
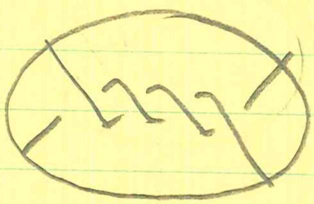
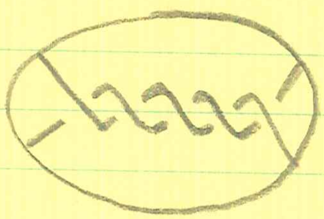
$$P_\infty^{(2)} = \left(P_0^{(1)}\right)^{\text{TWIST}} = A^{-1}A - A^{-3}A^{-1}$$

$$= 1 - A^{-4}$$

WHERE $\langle T_1 \rangle = \langle$  $\rangle = \underset{\underset{A}{\shortparallel}}{P_0^{(1)}} \langle$  $\rangle + \underset{\underset{A^{-1}}{\shortparallel}}{P_\infty^{(1)}} \langle$  $\rangle$

$A \longleftarrow \text{PART (a)} \longrightarrow A^{-1}$

IN GENERAL

	P_0	P_1
	A^2	$1-A^{-4}$
	A^3	$A-A^{-3}(1-A^{-4})$ $= \underbrace{A-A^{-3}+A^{-7}}$
	A^4	$A^2-A^{-3}(\overline{A-A^{-3}+A^{-7}})$ $=A^2-A^{-2}+A^{-6}-A^{-10}$
	A^5	$A^3-A^{-3}(A^2-A^{-2}+A^{-6}-A^{-10})$ $=A^3-A^{-1}+A^{-5}-A^{-9}+A^{-13}$

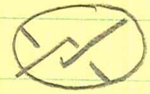
REASON: $T_{i+1} = (T_i)^{\text{TWIST}}$

(d) $\langle K_1 \rangle = \langle$  $\rangle$

$$= A^5(-A^{-2}-A^{+2}) + A^3-A^{-1}+A^{-5}-A^{-9}+A^{-13}$$

$$= \cancel{-A^3} - A^7 + \cancel{A^3} - A^{-1} + A^{-5} - A^{-9} + A^{-13}$$

$$= A^{-13}-A^{-9}+A^{-5}-A^{-1}-A^7$$

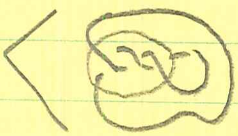
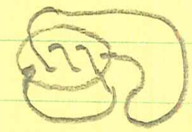
(e) WE'LL USE T_3 AND 

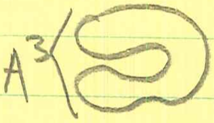
$\langle$  $\rangle = A^{-1}\langle$  $\rangle + A^{+1}\langle$  $\rangle$

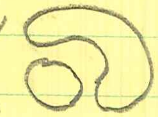
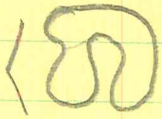
$= A^{-2}\langle$  $\rangle + \langle$  $\rangle + \langle$  $\rangle + A^{2}\langle$  $\rangle$

$= A^{-2}\langle$  $\rangle + (2-1-A^{4})\langle$  $\rangle$

$= A^{-2}e_0 + (1-A^4)e_\infty$

$\langle K_2\rangle = A^{-2}\langle$  $\rangle + (1-A^4)\langle$  $\rangle$

$= A^{-2}\Big(A^3\langle$  $\rangle + (A-A^{-3}+A^{-7})\langle$  $\rangle\Big)$

$+ (1-A^4)\Big(A^3\langle$  $\rangle + (A-A^{-3}+A^{-7})\langle$  $\rangle\Big)$

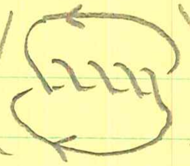
$$= A + (A-A^{-3}+A^{-7})(-A^{-2}-A^{2})A^{-2}$$

$$+ (1-A^4)(-A^{-2}-A^2)A^3 + (A-A^{-3}+A^{-7})(1-A^4)$$

$$= -A^5+A-A^{-3}+A^{-7}-A^{-3}+2A+(-A^{-2}-A^2)(A^{-9}-A^{-5}+A^{-1}+A^3-A^7)$$

$$= A^{-7}-2A^{-3}+3A-A^5-A^{-11}+A^{-7}-A^{-3}-A+A^5$$
$$-A^{-7}+A^{-3}-A-A^5+A^9$$

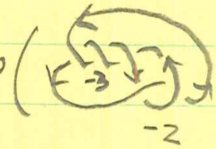
$$= -A^{-11}+A^{-7}-2A^{-3}+A-A^5+A^9$$

(f) $X(K_1) = (-A^3)^{-5} \langle K_1 \rangle$ $\qquad w\Big($  $\Big) = 5$

$= -A^{-15}\left(A^{-13} - A^{-9} + A^{-5} - A^{-1} - A^{7}\right)$

$= -A^{-28} + A^{-24} - A^{-20} + A^{-16} + A^{-8}$

$V_{K_1}(t) = -t^7 + t^6 - t^5 + t^4 + t^2$

$X(K_2) = (-A^3)^5 \langle K_2 \rangle$ $\qquad w\Big($  $\Big) = -5$

$= -A^{15}\left(-A^{-11} + A^{-7} - 2A^{-3} + A - A^5 + A^9\right)$

$= +A^4 - A^8 + 2A^{12} - A^{16} + A^{20} - A^{24}$

$V_{K_2}(t) = t^{-1} - t^{-2} + 2t^{-3} - t^{-4} + t^{-5} - t^{-6}$

$V_{K_1} \neq V_{K_2}$: $0 \neq V_{K_1}(t) \in \mathbb{Z}[t]$ WHILE $0 \neq V_{K_2} \in \mathbb{Z}[t^{-1}]$!

IT FOLLOWS THAT K_1 AND K_2 ARE DISTINCT KNOTS.