Math 427/527: algebraic topology

Homework 1, due Friday February 7 by 5:00 pm.

1. Let FinSet be the category of finite sets and let $\mathsf{Alg}_{\mathbf{k}}$ be the category of algebras over a field $\mathbf{k}$.

(i) Given a set $S = \{x_1, \ldots, x_n\}$, consider the algebra of all polynomials in $x_1, \ldots, x_n$ with coefficients in $\mathbf{k}$. Show that this construction defines a functor $\mathsf{FinSet} \to \mathsf{Alg}_{\mathbf{k}}$.

(ii) Give the definition of a monoidal functor.

(iii) Check that $(\mathsf{FinSet}, \sqcup, \varnothing)$ and $(\mathsf{Alg}_{\mathbf{k}}, \otimes, \mathbf{k})$ are monoidal categories, and show that the functor defined in (i) is a monoidal functor with respect to these monoidal structures.

2. Poincaré was first to realize that homology was weaker than homotopy, and he revised his now-famous conjecture accordingly.

(i) Show that the universal cover of the special orthogonal group $SO(3)$ may be identified with the unit quaternions, and is therefore homeomorphic to the three-sphere S^3.

(ii) Let I be the icosahedral group. Define an action of I on $SO(3)$, and determine the group $\tilde{I}$ acting on S^3, associated with the cover in part (i).

(iii) Show that $\tilde{I}$ is a perfect group, and conclude that there exists a topological space, locally homeomorphic to $\mathbb{R}^3$, that has non-trivial fundamental group but trivial first homology (assuming that this latter group is the abelianization of the former).

3. If X and Y are CW complexes, with characteristic maps Φ_α and Ψ_β, respectively, prove that $X \times_c Y$ is a CW complex with characteristic maps $\Phi_\alpha \times \Psi_\beta$.

4. Fix a category $\mathcal{C}$.

(i) Prove that if $f, g \in \mathrm{ar}(\mathcal{C})$ are composable (that is, composable morphisms in $\mathcal{C}$) such that $g \circ f$ and g are isomorphisms, then f is an isomorphism as well.

(ii) Prove that if $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ is a sequence of morphisms in $\mathcal{C}$ such that $g \circ f$ and $h \circ g$ are isomorphisms, then g (and therefore f and h) must be isomorphisms.

(iii) Now let $\mathcal{C}$ be Top, the category of topological spaces. Give the definition of a deformation retract and, using part (ii), show that a deformation retract is a homotopy equivalence.

5. Let X be a CW complex. The *unreduced* suspension SX of X is the quotient space $X \times I/{\sim}$ where $(x, 0) \sim (y, 0)$ for all $x, y \in X$ and $(x, 1) \sim (y, 1)$ for all $x, y \in X$. The *reduced* suspension then is just the usual the suspension: $\Sigma X = X \wedge S^1$. By factoring a choice of map $X \times I \to \Sigma X$ through SX, show that SX is homotopy equivalent to ΣX. (You may find it useful to consult Proposition 0.17 in Hatcher.)