Math 427/527: algebraic topology

Homework 3, due Friday March 27 by 5:00 pm.

1. Consider the function $f\colon [0,1] \to [0,1]$ given by $f(x) = 3x^2 - 2x^3$. The cell structure on $[0,1]$ (consisting of two 0-cells and one 1-cell) induces a cell structure on $[0,1]^n$; call this the *obvious* cell structure on the hypercube.

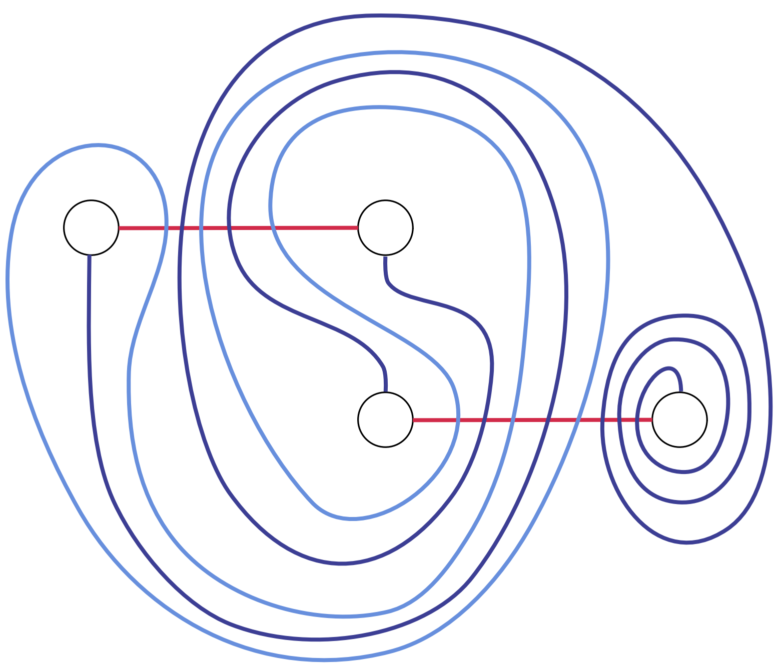
(i) Check that f is a Morse function on $[0,1]$.

(ii) Define $F(x_1, x_2, x_3) = f(x_1) + f(x_2) + f(x_3)$ for $x_i \in [0,1]$ and show that F is Morse.

(iii) Show that $-\nabla F$ induces the obvious cell structure on $[0,1]^3$.

(iv) Give the definition of a 2-manifold with corners, and give an explicit description of the set of flow lines of $-\nabla F$ as a 2-manifold with corners.

2. Let $f\colon M \to [0,3]$ be a self-indexing Morse function on a space M, which has a unique maximum value and a unique minimum value. The picture bellow describes $f^{-1}(\frac{3}{2})$, which is a genus 2 surface obtained from S^2 (the page, compactified) after identifying each pair of holes at the same height (without any twisting):



You should assume that the two red curves consist of points flowing to index 1 critical points (along $-\nabla f$) and the two blue curves consist of points flowing to the index 2 critical points (along ∇f).

(i) Using the description above, calculate the cellular homology groups $H_i(M; \mathbb{Z})$.

(ii) Calculate $\pi_1(M)$, noting that each blue curve describes a 2-cell attaching map.

(iii) Find a surjection from $\pi_1(M)$ to A_5.

3. Determine $H_1(\widehat{X};\mathbf{k})$ and $H_0(\widehat{X};\mathbf{k})$ as $\mathbf{k}[t,t^{-1}]$-modules, where X is the Klein bottle with infinite cyclic cover $\widehat{X}$ and $\mathbf{k}$ is any field.

4. Let X be the complement of a knot $K\colon S^1 \hookrightarrow S^3$, and let $\widehat{X}$ be the infinite cyclic cover determined by the Hurewicz map. Recall that if $p\colon \widehat{X} \to X$ is the associated covering map, then

$$p_*\colon \pi_1\widehat{X} \to \pi_1 X$$

is a homeomorphism with image C isomorphic to the commutator subgroup $[\pi_1 X, \pi_1 X]$ in $\pi_1 X$. The key observation for this problem is that p_* induces an isomorphism

$$\bar{p}_*\colon H_1(\widehat{X};\mathbb{Z}) \to \frac{C}{[C,C]}$$

Write Λ for the ring of Laurent polynomials $\mathbb{Z}[t,t^{-1}]$.

(i) Suppose $c \in C$ and let $x \in \pi_1 X$ be an element sent to a generator under abelianization. Define $\mathbf{t}[c] = [xcx^{-1}]$ where $[\cdot]$ denotes the coset in $C/[C,C]$. Show that $\mathbf{t}$ is a well-defined automorphism of $C/[C,C]$.

(ii) Let t be the generator of Λ acting on $H_1(\widehat{X};\mathbb{Z})$. Show that, up to appropriate choices of generators, $\bar{p}_* \circ t = \mathbf{t} \circ \bar{p}_*$. (This promotes $\bar{p}_*$ to an isomorphism of Λ-modules.)

(iii) Show that C is generated by all words of the form $x^k g_i^{\pm 1} x^{-k}$ (x as above) given a generating set $x, g_1 \ldots, g_n$ for $\pi_1 X$ where each $g_i \in C$, so that the substitution

$$\pm\mathbf{t}^k \gamma_i \longleftrightarrow x^k g_i^{\pm 1} x^{-k}$$

gives rise to a Λ-module presentation for $C/[C,C]$.

(iv) Let K be the trefoil (as seen in class) so that

$$\pi_1 X \cong \langle x, y | xyx = yxy \rangle$$

Setting $a = yx^{-1}$, and following the strategy suggested above, determine Λ-module structure on $C/[C,C]$.

5. As seen in class, any $K\colon S^1 \hookrightarrow S^3$ bounds an orientable surface F. Let X be the complement of the trefoil knot, and consider the 2-fold cover $Y \to X$ obtained by sending $\pi_1 X$ onto $\mathbb{Z}/2\mathbb{Z}$. Using the Mayer-Vietoris sequence, calculate $H_1(Y;\mathbb{Z})$.