Math 309: Introduction to knot theory

Assignment 1, due September 25 by 11:59 pm.

Exercises.

1. Recall, from class, that for a pair of relatively prime positive integers p and q we can form a knot with parametrization

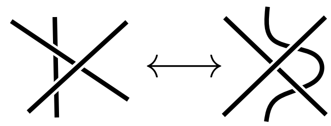
$$\begin{aligned} x(t) &= (2+\cos(pt))\cos(qt) \\ y(t) &= (2+\cos(pt))\sin(qt) \\ z(t) &= \sin(pt) \end{aligned}$$

where $0 \leq t \leq 2\pi$.

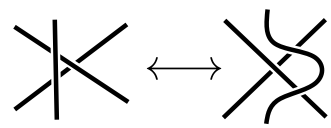
(a) Consider the knot obtained in the case $p = 2, q = 3$, and consider the result of projecting this to the xy-plane and to the yz-plane. Do both these projections give rise to knot diagrams? Why or why not?

(b) Find, with proof, values of p and q for which this paramatrization gives the knot 5_1.

2. Figure 1.24 in §1.3 of Adams depicts the following two distinct Reidemeister moves of type III:



(1)



(2)

Using the Reidemeister moves of type I and II and the move (1), deduce the move (2). (It is also possible to show that the move (1) follows from the type I and II moves together with (2)).

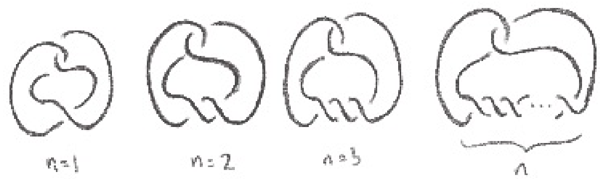
3. Links are introduced in Adams §1.4. Show that the following diagrams describe the same link:

In your solution, explicitly identify where you made use of a Reidemeister Type III move.

4. Decide the 3-colourability of each of the knots 7_5, 7_6, 7_7: In each case, either give a valid 3-colouring or prove that no 3-colouring exists.

Problems

5. The following infinite family of knots are referred to as *twist knots*:



(a) Check that the case $n = 1$ is the knot 3_1; and that the case $n = 4$ is the knot 6_1.

(b) Prove that a twist knot is 3-colourable if and only if $n = 3k + 1$. (Recall that an *if and only if* statement requires that you check two implications!)

6. Using the notation from the table in Adams, the figure eight knot is the knot 4_1 and the trefoil knot is the knot 3_1.

(a) A 5-colouring of a knot diagram is a choice of label from the set $\{0, 1, 2, 3, 4\}$ (these are the five "colours") for each of the strands so that at least two colours are used and at each crossing, if z is the label on the overstrand, the equation

$$x + y - 2z = 0 \text{ modulo } 5$$

holds. Show that the existence/non-existence of such a colouring is preserved under each of the Reidemeister moves.

(b) Show that the figure eight knot is 5-colourable.

(c) Using material that we have seen in the course and homework to this point, prove that the unknot, the trefoil, and the figure eight knot are mutually distinct.