Math 309: Introduction to knot theory

Assignment 3, due Wednesday October 14 by 11:59 pm.

Exercises.

1. Adams defines the bridge number of a knot K as the smallest number of maximal overpasses in any knot diagram for K. In class, we proved that the bridge number is equal to the smallest number of local maxima over all possible embeddings of K in $\mathbb{R}^3$.

(a) Provide definitions for "maximal overpass" and "local maximum" in this context. (This an exercise in reading Adams' textbook and the lecture notes).

(b) Draw a projection of the figure eight knot that has exactly two local maxima and apply the method of proof for this theorem to find a projection of the figure eight with exactly two maximal overpasses.

2. (Adams 3.10) Prove that if a knot has bridge number 1, then it is the unknot. What does this tell you about the bridge number of the figure eight knot?

3. Prove that that there are infinitely many distinct tangles whose closure is the unknot. This shows that the continued fraction associated with a rational tangle T is not an invariant of the knot obtained as the closure of T.

4. (Adams 2.16) Characterize the Conway notation of the rational tangles whose closure is a 2-component link.

Problems

5. In this problem we will play with continued fractions. We will use the Conway notation $[a_1, \ldots, a_n]$ to also denote the continued fraction

$$a_n + \cfrac{1}{a_{n-1} + \cfrac{1}{\ddots + \frac{1}{a_1}}}$$

(a) Express $\frac{25}{11}$ as a continued fraction $[a_1, \ldots, a_n]$ such that $a_n < 25/11$. Is it possible to make it so that every a_i in the continued fraction is a positive integer?

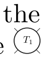
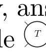
(b) Express $\frac{25}{11}$ as a continued fraction $[a_1, \ldots, a_n]$ such that $a_n > 25/11$. Is it possible to make it so that every a_i in the continued fraction is a positive integer?

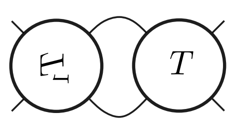
(c) Draw the rational tangle associated with each of the continued fractions you found in the previous section. Is either of these tangles alternating?

(d) By manipulating the tangle diagrams, show that they are equivalent. (You do not need to list every Reidemeister moves, just show your steps.)

6. Recall the definition of "addition" and "multiplication" of 4-ended tangles in §2.3 of Adams.

(a) (Adams 2.21) Is there an additive identity for 4-ended tangles? Support your answer with an argument (in other words, a proof).

(b) (Adams 2.22) Is there a multiplicative identity for 4-ended tangles? You should consider multiplication on the left and multiplication on the right separately. More precisely, answer the following two questions: (1) is there a special tangle  such that, for every other tangle , the product



is equal to ; (2) same question, with the roles of T and T_1 reversed

(c) (Adams 2.26) Provide a solution to Exercise 2.26 in Adams. (Hint: doing Exercise 2.25 first should help.)