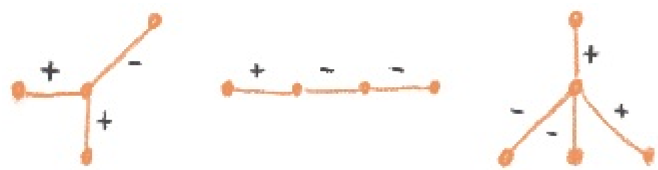
Math 309: Introduction to knot theory

Assignment 4, due Wednesday October 28 by 11:59 pm.

Exercises.

1. Consider the following recipe, which sets up a one-two-one correspondence between **twisted unknots** (left) and **signed trees** (right):

(a) For each of the following signed trees, find the associated twisted unknot and calculate its bracket polynomial:



(b) Explain how to quickly compute the bracket polynomial of a twisted unknot. (Quickly means linearly in the number of crossings.)

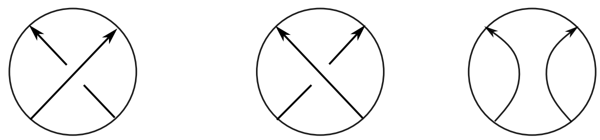
2. Compute the bracket polynomial and writhe of the figure eight knot, 4_1.

3. Compute the bracket polynomial of the knot 5_2 **using a tree of resolutions** and compute the writhe of 5_2 (after choosing an orientation on 5_2, of course). (Hint: $2^5 = 32$, so you should probably use the result you found for exercise 1).

4. Exercise 6.8 of Adams' book is to show that the Jones polynomial satisfies the following "skein relation"

$$t^{-1}V(L_+) - tV(L_-) = (t^{1/2} - t^{-1/2})V(L_0), \tag{1}$$

where L_+, L_-, L_0 are oriented diagrams of links that are the same everywhere except for the interior of a circle, inside of which they are as follows:



L_+ $\qquad$ L_- $\qquad$ L_0

Show that if a polynomial invariant of links that assigns to a link L the polynomial $p_L(t) \in \mathbb{Z}[t, t^{-1}]$ satisfies the skein relation (1), and takes the value $p_{\text{unknot}}(t) = t^{-1} + t$, then

$$p_{L^*}(t) = p_L(t^{-1})$$

where K^* is the mirror of K.

Problems

5. Recall the ring of Laurent polynomials $\mathbb{Z}[t, t^{-1}]$ is defined to be the set of finite sums of elements of the form

$$c_i t^i$$

where $i \in \mathbb{Z}$ and $c_i \in \mathbb{Z}$, together with the natural operations of addition and multiplication.

(a) Check that $(t^{-2} + t^{-4} + t^{-6} + \cdots)$ is an inverse for the element $(t^2 - 1)$. Is $t^2 - 1$ invertible in $\mathbb{Z}[t, t^{-1}]$?

(b) To proceed more systematically, let $A, B \in \mathbb{Z}[t, t^{-1}]$ be arbitrary elements, so there are finite sets of integers I and J, and for every $i \in I$ and for every $j \in J$, there are integers a_i and b_j such that

$$A = \sum_{i \in I} a_i t^i \qquad B = \sum_{j \in J} b_j t^j$$

The product AB is an element $C \in \mathbb{Z}[t, t^{-1}]$, so there is a finite set $K \subset \mathbb{Z}$ and integers c_k such that

$$C = \sum_{k \in K} c_k t^k$$

Express the coefficients c_k in terms of the coefficients a_i and b_j.

(c) Use part (b) to prove that if $AB = 1$, then $A = \pm t^n$, for some $n \in \mathbb{Z}$.

6. Compute the Jones polynomial of T_n, the n^{th} twist knot as defined in assignment 1. (Hint: consider separately the cases "n is even" and "n is odd". You will probably want to use the skein relation mentioned in exercise 4 and have to solve a recursive formula.)