Lecture #10

Last time we saw that the honeycomb tiling is isohedral, it is periodic, and it is monohedral.

\[ P(\vec{a}, \vec{b}) = \]

\[ \text{Translate piece by } \vec{b} \]

\[ \text{Translate piece by } \vec{a} \]

So rearranging these pieces of the patch \( P(\vec{a}, \vec{b}) \) [where the pieces can be viewed as tiles \( \vartriangle \) and \( \vartriangle \)] gives a new patch, which happens to be \( T = \vartriangle \).

This re-establishes the isohedral property, obtained by translation along \( n\vec{a} + m\vec{b} \) for \( n, m \in \mathbb{Z} \) (why?)

This patch obtained from \( P(\vec{a}, \vec{b}) \) could be more complicated.
Call the result of this construction \( \hat{P}(\alpha, b) \).

Translated pieces clustered along \( \ell(\alpha, 2) \) direction (for some \( x \in \mathbb{R}^2 \)).

Given \( \hat{P}(\alpha, b) \) we can always build this \( \hat{P}(\alpha, b) \) if we just use our trick for building tiles (from a square tile) that admit a monohedral tiling.

That is:

Using this idea, we can prove:

"Theorem" Any periodic monohedral tiling is isohedral.

"Proof" Fix a periodic monohedral tiling \( T \); tile \( T \).
It will be helpful to have a schematic.
Consider the standard tiling by unit squares, but make a "bad" choice for \( P(\alpha, b) \).
WE KNOW THAT $P(\hat{x}, \hat{y})$ TILES THE PLANE, AND WE HAVE CHECKED THAT THE PATCH $\hat{P}(\hat{x}, \hat{y})$ (A UNION OF CONGRUENT TILES) DOES TO.

NOW USE THE FACT THAT EVERY $T_1, T_2 \in \hat{P}(\hat{x}, \hat{y})$ ARE A CONGRUENT PAIR; LET $g: \mathbb{R}^2 \to \mathbb{R}^2$ BE A CONGRUENCE SO THAT $g(T_1) = T_2$.

FOR EXAMPLE:

$$ g \text{ rotates by } 90^\circ \text{ about this point} $$

(Note: There may be lots of choices for $g$!!)

By definition $g$ extends to the whole plane, it takes $\hat{P}$ to a congruent copy of $\hat{P}$, and $P$ to a congruent copy of $P$.

So $\hat{g}$ is isohedral: if $T \in \hat{P}$ and $T' \in \hat{P}'$, first translate $\hat{g} : \hat{P}' \to \hat{P}$ (a congruence) then compose with $g: \mathbb{R}^2 \to \mathbb{R}^2$, so that $g(\hat{g}(T')) = T'$. WHERE IS THE ISSUE?
Last time we considered \( P(\alpha, \beta) \) in a periodic monoclinic tiling, and constructed a patch \( \hat{P}(\alpha, \beta) \) consisting of tiles.

\[
P(\alpha, \beta) \quad \rightarrow \quad \hat{P}(\alpha, \beta)
\]

Not unique! Can choose different translations.

Also not unique!

\[
\hat{P}(\alpha, \beta) \quad \rightarrow \quad \hat{P}_{\text{alt}}(\alpha, \beta)
\]

Using a choice of \( \hat{P}(\alpha, \beta) \) for some \( \gamma \), we first argued that, translating by \( \mathbf{m} + \mathbf{n} \mathbf{b} \), any \( \hat{P}' \subset \gamma \) could be moved to \( \hat{P} \subset \gamma \).

\[
\hat{P} \quad \rightarrow \quad \hat{P}'
\]

\[
t: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \quad X \mapsto X + (\mathbf{m} + \mathbf{n} \mathbf{b})
\]

So \( X \in \hat{P} \) is \( X' + (\mathbf{m} + \mathbf{n} \mathbf{b}) \) for \( X' \in \hat{P}' \).

This is a symmetry of \( \gamma \) coming from periodicity.

As a result, if \( \hat{P}, \hat{P}' \) are distinct, any two tilings \( T \hat{P} \) and \( T \hat{P}' \) are equivalent in \( \gamma \).

Recall that \( \gamma \) is isomorphral if all tiles of \( \gamma \) are equivalent.
Next, we considered $T_1, T_2 \in \hat{P} \subset \hat{J}$. Using the fact that $\hat{J}$ is monohedral, we know that $T_1$ and $T_2$ are congruent, so it is possible to pick

$$g : \mathbb{R}^2 \to \mathbb{R}^2$$

$$T_1 \mapsto T_2$$

This $g$ is a congruence of the plane, but here is the plan in the argument:

⚠️ It may not be possible to choose $g \in S(\hat{J})$.

For the square thing it is possible:

If we choose $\hat{P}(\alpha, \beta) = \begin{array}{c} \end{array}$

Then $(T_{12}, \pi_1, \pi_2)$ will give congruences that fix $\hat{P}$, and hence do extend to $S(\hat{J})$.

(We could also see this by picking $\hat{P} = T_0$.)

Obs if $\hat{J}$ is monohedral and periodic and there is a choice $\hat{P}$ so that $T_1, T_2 \in \hat{P}$ are equivalent by an element of $S(\hat{P})$, then $\hat{J}$ is isohedral.
How things break down (an example): $T = \begin{array}{c|c}
\cline{2-3}
& 1 \\
\hline
\end{array}$

$P(\alpha, \beta) = \begin{array}{c|c|c}
\cline{2-3}
& & \\
\hline
\end{array}$

$T_1$, $T_2$

This example is actually 2-isohedral:

WE SEE PATCHES LIKE

\[
\begin{align*}
\text{4 equivalent tiles} & \quad \text{2 equivalent tiles}
\end{align*}
\]

DEF $T_1, T_2$ are in the same "transitivity class" if there is a $g \in S(\mathcal{G})$ so that $g T_1 = T_2$. And if $\mathcal{G}$ has $p$ transitivity classes it is "p-isohedral".

RMK: $P$ is the number of tiles in $\hat{P}$, then $\mathcal{G}$ has at most $|P|$ transitivity classes.