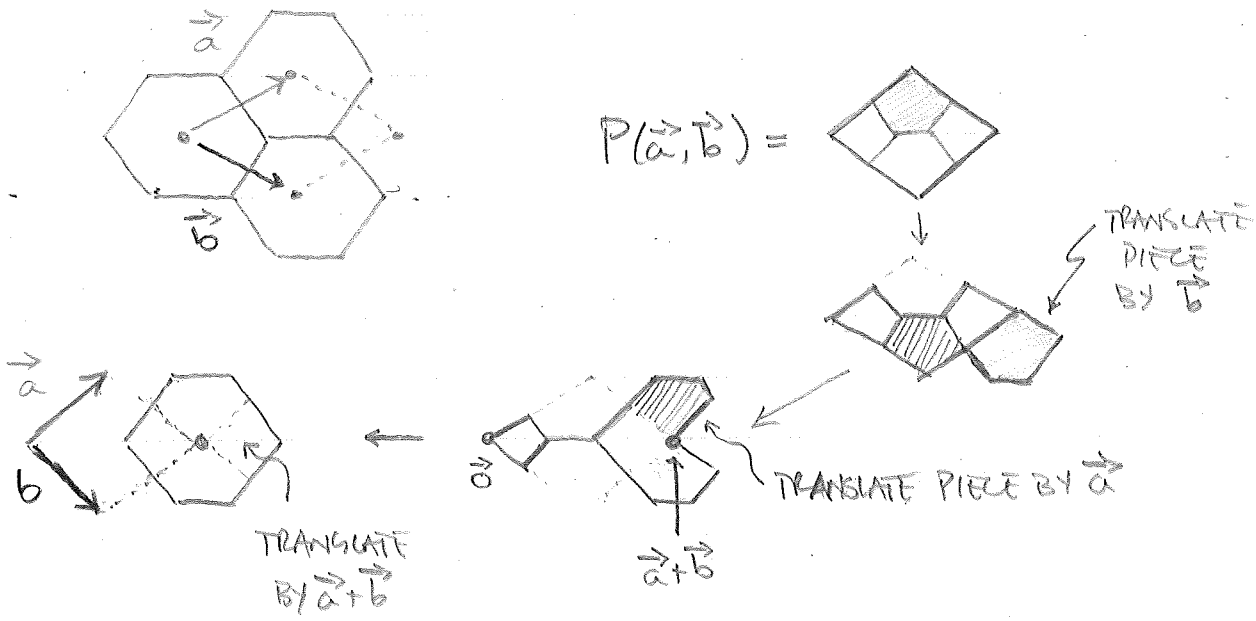
LECTURE #10

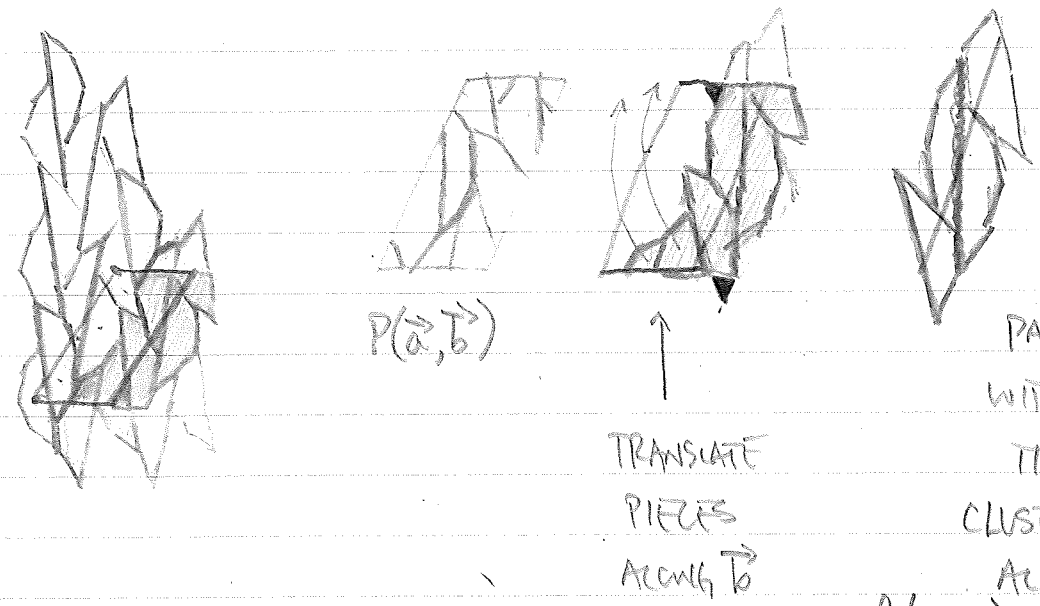
LAST TIME WE SAW THAT THE HONEYCOMB TILING IS ISOHEDRAL. IT IS PERIODIC, AND IT IS MONOHEDRAL.



SO REARANGING THESE PIECES OF THE PATCH $P(\vec{a},\vec{b})$ [WHERE THE PIECES CAN BE VIEWED AS TILES ] GIVES A NEW PATCH, WHICH HAPPENS TO BE $T =$ 

<u>RMK</u> THIS RE-ESTABLISHES THE ISOHEDRALITY, OBTAINED BY TRANSLATION ALONG $n\vec{a} + m\vec{b}$ FOR $n, m \in \mathbb{Z}$ (WHY?)

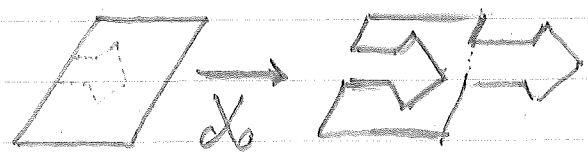
THIS PATCH OBTAINED FROM $P(\vec{a},\vec{b})$ COULD BE MORE COMPLICATED:



CALL THE RESULT OF THIS CONSTRUCTION $\hat{P}(\vec{a},\vec{b})$. $\Longrightarrow$ (FOR SOME $X \in \mathbb{R}^2$).

GIVEN $P(\vec{a},\vec{b})$ WE CAN ALWAYS BUILD THIS $\hat{P}(\vec{a},\vec{b})$; WE JUST USED OUR TRICK FOR BUILDING TILES (FROM A SQUARE TILE) THAT ADMIT A MONOHEDRAL TILING.

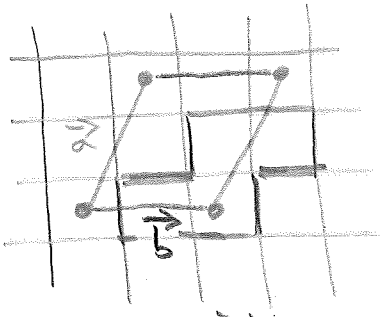
THAT IS:



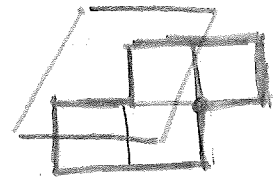
USING THIS IDEA, WE CAN PROVE:

"THEOREM" ANY <u>PERIODIC</u> <u>MONOHEDRAL</u> TILING IS <u>ISOHEDRAL</u>.

"PROOF" FIX A PERIODIC MONOHEDRAL TILING $\mathcal{T}$; TILE T.
IT WILL BE HELPFUL TO MAKE A SCHEMATIC.
CONSIDER THE STANDARD TILING BY UNIT SQUARES, BUT MAKE A "BAD" CHOICE FOR $P(\vec{a},\vec{b})$:



$P(\vec{a},\vec{b})$



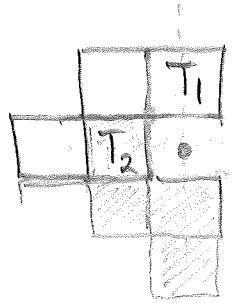
$\hat{P}(\vec{a},\vec{b})$

WE KNOW THAT $P(\vec{a},\vec{b})$ TILES THE PLANE, AND WE HAVE CHECKED THAT THE PATCH $\hat{P}(\vec{a},\vec{b})$ (A UNION OF CONGRUENT TILES) DOES TO.

NOW USE THE FACT THAT EVERY $T_1, T_2 \in \hat{P}(\vec{a},\vec{b})$ ARE A CONGRUENT PAIR; LET $g: \mathbb{R}^2 \to \mathbb{R}^2$ BE A CONGRUENCE SO THAT

$$g(T_1) = T_2$$

FOR EXAMPLE:



g ROTATES BY $90°$ ABOUT THIS POINT

(NOTE: THERE MAY BE LOTS OF CHOICES FOR g !!)

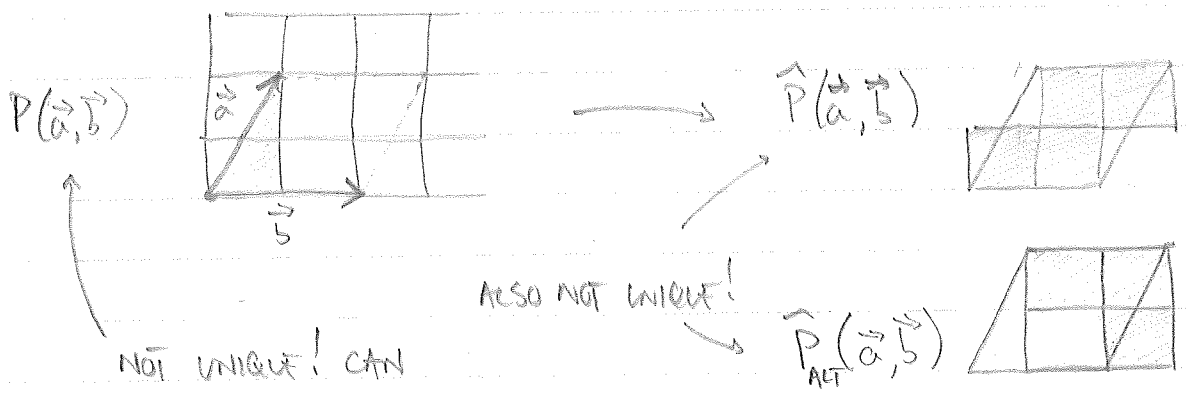
BY DEFINITION g EXTENDS TO THE WHOLE PLANE. IT TAKES $\hat{P}$ TO A CONGRUENT COPY OF $\hat{P}$; AND P TO A CONGRUENT COPY OF P.

SO $\mathcal{T}$ IS ISOHEDRAL: IF $T \in \hat{P}$ AND $T' \in \hat{P}'$, FIRST TRANSLATE $t: \hat{P}' \mapsto \hat{P}$ (A CONGRUENCE) THEN COMPOSE WITH $g: \mathbb{R}^2 \to \mathbb{R}^2$ SO THAT $g(\underbrace{tT'}_{T_1}) = \underbrace{T}_{T_2}$.

$\boxed{?}$ WHERE IS THE ISSUE?

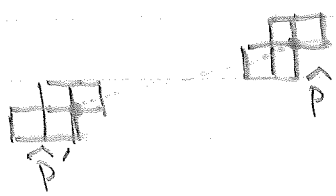
LECTURE # 11

LAST TIME WE CONSIDERED $P(\vec{a},\vec{b})$ IN A PERIODIC MONOHEDRAL TILING, AND CONSTRUCTED A PATCH $\hat{P}(\vec{a},\vec{b})$ CONSISTING OF TILES



NOT UNIQUE! CAN CHOOSE DIFFERENT TRANSLATIONS

USING A CHOICE OF $\hat{P}(\vec{a},\vec{b})$ FOR SOME $\mathcal{T}$, WE FIRST ARGUED THAT, TRANSLATING BY $m\vec{a}+n\vec{b}$, ANY $\hat{P}' \subset \mathcal{T}$ COULD BE MOVED TO $\hat{P} \subset \mathcal{T}$.



$$t: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$X \longmapsto X + (n\vec{a} + m\vec{b})$$

SO $X \in \hat{P}$ IS $X' + (n\vec{a}+m\vec{b})$ FOR $X' \in \hat{P}'$.

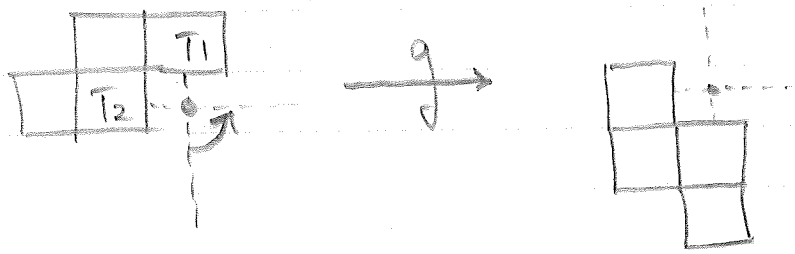
THIS IS A SYMMETRY OF $\mathcal{T}$ COMMING FROM PERIODICITY.

AS A RESULT, IF $\hat{P}, \hat{P}'$ ARE DISTINCT, ANY TWO TILES $T \in \hat{P}$ AND $T' \in \hat{P}'$ ARE EQUIVALENT IN $\mathcal{T}$.

RECALL THAT $\mathcal{T}$ IS ISOHEDRAL IF <u>ALL</u> TILES OF $\mathcal{T}$ ARE EQUIVALENT.

NEXT, WE CONSIDERED $T_1, T_2 \in \hat{P} \subset \mathcal{T}$. USING THE FACT THAT $\mathcal{T}$ IS MONOHEDRAL, WE KNEW THAT T_1 AND T_2 ARE CONGRUENT, SO IT IS POSSIBLE TO PICK

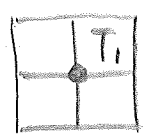
$$g : \mathbb{R}^2 \to \mathbb{R}^2$$
$$T_1 \longmapsto T_2$$



THIS g IS A CONGRUENCE OF THE PLANE, BUT HERE IS THE FLAW IN THE ARGUMENT:

⚠ IT MAY NOT BE POSSIBLE TO CHOOSE $g \in S(\mathcal{T})$.

FOR THE SQUARE TILING IT IS POSSIBLE:

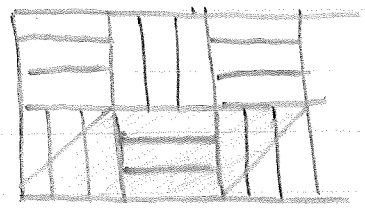
IF WE CHOOSE $\hat{P}(\vec{a}, \vec{b}) =$ 

THEN $r_{\pi/2}, r_{\pi}, r_{3\pi/2}$ WILL GIVE CONGRUENCES THAT FIX $\hat{P}$, AND HENCE <u>DO</u> EXTEND TO $S(\mathcal{T})$!

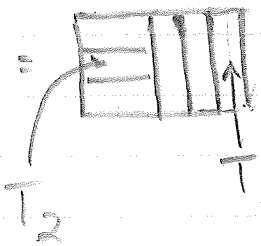
(WE COULD ALSO SEE THIS BY PICKING $\hat{P} = T$!).

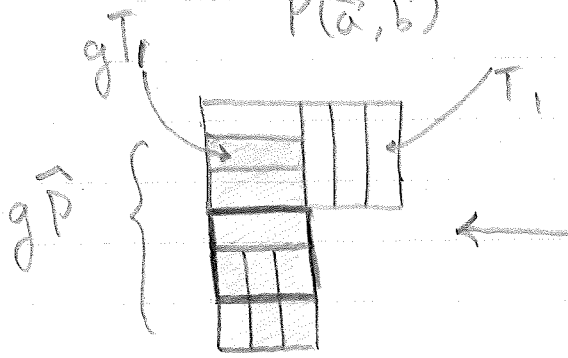
<u>OBS</u> IF $\mathcal{T}$ IS MONOHEDRAL AND PERIODIC AND THERE IS A CHOICE $\hat{P}$ SO THAT $T_1, T_2 \in \hat{P}$ ARE EQUIVALENT BY AN ELEMENT OF $S(\hat{P})$, THEN $\mathcal{T}$ IS ISOHEDRAL.

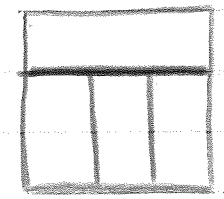
HOW THINGS BREAK DOWN (AN EXAMPLE): $T =$ 

$\mathcal{T} =$ 

$P(\vec{a},\vec{b})$

$\hat{P}(\vec{a},\vec{b}) =$ 

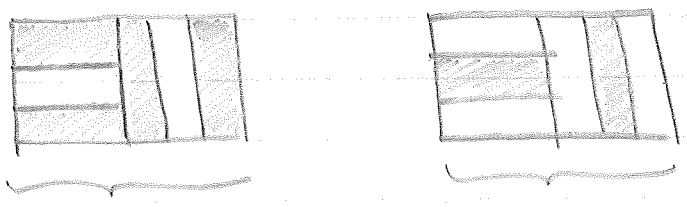




NOT A PATH OF TILES!

(WE SEE PATCHES LIKE  AND )

THIS EXAMPLE IS ACTUALLY 2-ISOHEDRAL:



4 EQUIVALENT TILES 2 EQUIVALENT TILES

<u>DEF</u> T_1, T_2 ARE IN THE SAME "TRANSITIVITY CLASS" IF THERE IS A $g \in S(\mathcal{T})$ SO THAT $gT_1 = T_2$; AND IF $\mathcal{T}$ HAS k TRANSITIVITY CLASSES IT IS "k-ISOHEDRAL"

<u>RMK</u> IF $|\hat{P}|$ IS THE NUMBER OF TILES IN $\hat{P}$, THEN $\mathcal{T}$ HAS AT <u>MOST</u> $|\hat{P}|$ TRANSITIVITY CLASSES.