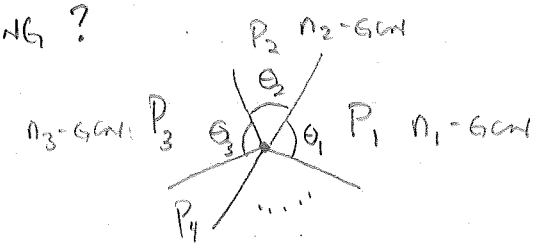
LECTURE # 13

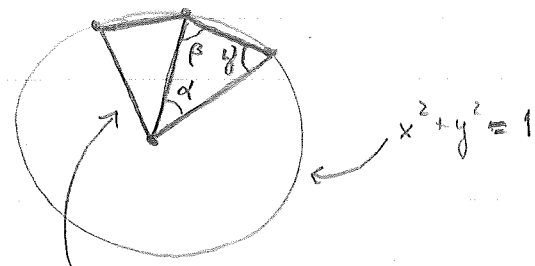
RECALL THAT THE BOUNDARY OF A POLYGON CONSISTS OF VERTICES (CORNERS) AND EDGES.

HOW DO A COLLECTION OF <u>REGULAR</u> POLYGONS MEET AT A VERTEX IN A TILING?



EACH $n_i \in \mathbb{Z}_{>2}$

GIVEN P_i, WHAT IS Θ_i?



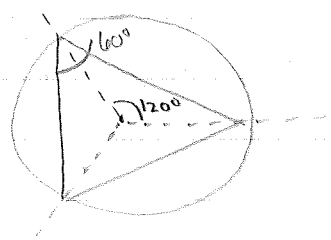
P_i DIVIDES INTO n_i TRIANGLES WITH $\beta = \gamma$.

* $\Theta_i = \beta + \gamma$
* $\alpha = \frac{1}{n} 2\pi$

$$\Rightarrow \quad \alpha + \beta + \gamma = \pi$$

$$\Theta_i = \beta + \gamma = \frac{n\pi}{n_i} - \frac{2\pi}{n_i} = \frac{n_i - 2}{n_i}\pi$$

EXP $(n=3)$



$$\Theta = \frac{n-2}{n}\pi = \frac{3-2}{3}\pi = \frac{1}{3}\pi$$

SO WHEN THE P_i FIT TOGETHER IN A TILING WE NEED

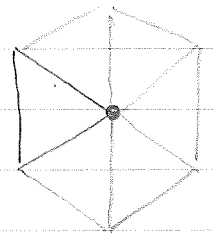
$$\sum_{i=1}^{r} \Theta_i = \sum_{i=1}^{r} \frac{n_i - 2}{n_i}\pi = 2\pi$$

THIS IS THE <u>SAME AS PREVIOUSLY</u>!

$$\boxed{\sum_{i=1}^{r} \frac{n_i - 2}{n_i} = 2}$$

WE'VE SEEN MONOHEDRAL TILINGS BY REGULAR POLYGONS...

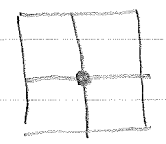
1. $n=3$



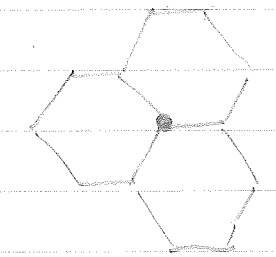
$$\frac{n-2}{n} = \frac{1}{3} \quad \& \quad 2 = 6\frac{1}{3}$$

6 EQUILATERAL TRIANGLES MEET AT A VERTEX

2. $n=4$ $\quad \frac{n-2}{n} = \frac{1}{2} \quad \& \quad 2 = 4\frac{1}{2}$



3. $n=6$ $\quad \frac{n-2}{n} = \frac{4}{6} = \frac{2}{3} \quad \& \quad 2 = 3 \cdot \frac{2}{3}$



WHY DID WE SKIP $n=5$? $\quad \frac{n-2}{n} = \frac{5-2}{5} = \frac{3}{5}$ BUT 2 DOES NOT DECOMPOSE INTO $\frac{3}{5}$ THS.

$$3\frac{3}{5} < 2 < 4\frac{3}{5}$$

TOO FEW 5-GONS $\qquad$ TOO MANY 5-GONS.

WHAT ABOUT $n>6$? WHAT HAPPENS FOR $n \gg 0$?

IN GENERAL, WE COULD ASK:

FOR A FIXED r, HOW MANY OF EACH n_i-GON P_{n_i} IS NEEDED?

WHICH r ARE POSSIBLE?

FOR EXAMPLE, FOR n_i FIXED ONLY $n = 3, 4, 6$ IS POSSIBLE.

WHEN WE START TO MIX AND MATCH, THINGS GET MORE INTERESTING. SOME EXAMPLES:

1. WHAT IF WE MIX SQUARES AND TRIANGLES? THEN

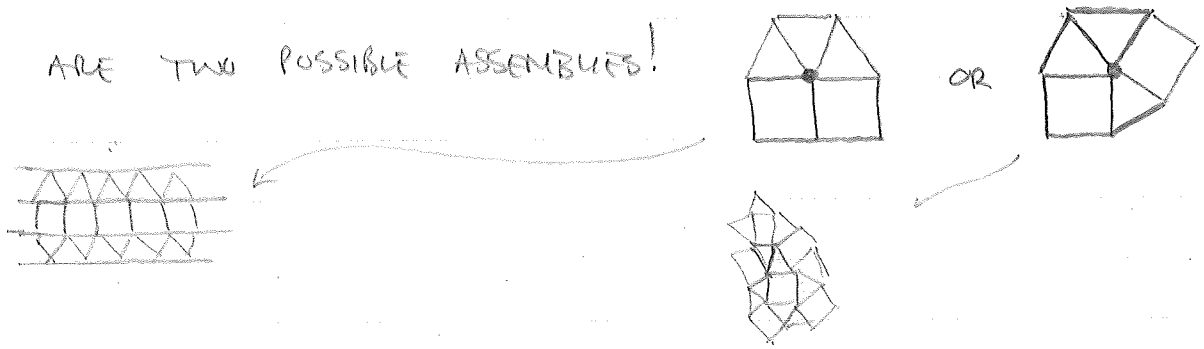
$$k \cdot \frac{3-2}{3} + l \cdot \frac{4-2}{4} = 2$$

$\#\triangle$ $\#\square$

$$k \cdot \frac{1}{3} + l \cdot \frac{1}{2} = 2$$

A COMBINATION THAT WORKS: $k = 3$, $l = 2$ (ARE THERE OTHERS?)

THERE ARE TWO POSSIBLE ASSEMBLIES!



2. MOST INTERESTING IS WHEN 3 DIFFERENT n-GONS INTERACT. WHAT TRIPLES $\{n_1, n_2, n_3\}$ ARE POSSIBLE?

A SURPRISING COMBINATION INVOLVES A 42-GON:

$$\frac{n_1-2}{n_1} + \frac{n_2-2}{n_2} + \frac{40}{42} = 2$$

$$42\left(n_2(n_1-2) + n_1(n_2-2)\right) + 40 n_1 n_2 = 2 \cdot 42 n_1 n_2$$

$$\vdots$$

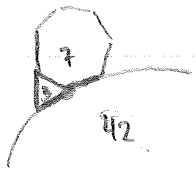
$$n_1 = 3, \; n_2 = 7$$

TRY THIS!

(SOLUTION NEXT PAGE)

CHECK

$$\boxed{\frac{1}{3} + \frac{5}{7} + \frac{40}{42} = 2}$$



HOW TO DEDUCE $n_1 = 3$, $n_2 = 7$ ...

$$42\left(n_2(n_1-2) + n_1(n_2-2)\right) + 40 n_1 n_2 = 2\cdot 42\, n_1 n_2$$

$$\cancel{2\cdot 42\, n_1 n_2} - 2\cdot 42\, n_2 - 2\cdot 42\, n_1 + 40 n_1 n_2 = \cancel{2\cdot 42\, n_1 n_2}$$

$$10\, n_1 n_2 = 21\,(n_1 + n_2)$$

NOTE THAT $10 = 5\cdot 2$ AND $21 = 7\cdot 3$, SO

① $n_1 n_2$ SHOULD BE (A MULTIPLE OF) 21
② $n_1 + n_2$ SHOULD BE (A MULTIPLE OF) 10

(THINK ABOUT PRIME FACTORS; THE COMMON MULTIPLE DIVIDES OUT.)

SO $n_1 n_2 = 21$ AND $n_1 + n_2 = 10$ GIVES $n_1 = 3$, $n_2 = 7$.