LECTURE # 13

RECALL THAT THE BOUNDARY OF A POLYGON CONSISTS OF VERTICES (CORNERS) AND EDGES.

HOW DO A COLLECTION OF REGULAR POLYGONS MEET AT A VERTEX IN A TILING?

\[
\begin{array}{c}
\text{P}_2 \quad n_2 \text{-gon} \\
\text{P}_3 \quad n_3 \text{-gon} \\
\text{P}_1 \quad n_1 \text{-gon}
\end{array}
\]

\[ \sum_{i=1}^{\infty} \eta_i \in \mathbb{Z}_{>2} \]

GIVEN \( P_i \), WHAT IS \( \Theta_i \)?

\[ \Theta_i = \beta_i + \gamma_i \]

\[ \alpha = \frac{1}{n} 2\pi \]

\[ \Rightarrow \alpha + \beta + \gamma = \pi \]

\[ \Theta_i = \beta_i + \gamma_i = \frac{n_i \pi}{\eta_i} - \frac{2\pi}{\eta_i} = \frac{\eta_i - 2}{\eta_i} \pi \]

\[ \text{EX} \quad (n = 3) \]

\[ \Theta = \frac{\eta - 2}{n} \pi = \frac{3 - 2}{3} \pi = \frac{1}{3} \pi \]

SO WHEN THE \( P_i \) FIT TOGETHER IN A TILING WE NEED

\[ \sum_{i=1}^{\infty} \Theta_i = \sum_{i=1}^{\infty} \frac{\eta_i - 2}{\eta_i} \pi = 2\pi \]

THIS IS THE SAME AS REQUIRING:

\[ \sum_{i=1}^{\infty} \frac{\eta_i - 2}{\eta_i} = 2 \]
We've seen monohedral tilings by regular polygons...

1. \( n = 3 \)  
   \[
   \frac{n-2}{n} = \frac{1}{3} \quad \therefore \quad 2 = 6 \cdot \frac{1}{3}
   \]
   (6 equilateral triangles meet at a vertex)

2. \( n = 4 \)  
   \[
   \frac{n-2}{n} = \frac{1}{2} \quad \therefore \quad 2 = \frac{4}{2}
   \]

3. \( n = 6 \)  
   \[
   \frac{n-2}{n} = \frac{4}{6} = \frac{2}{3} \quad \therefore \quad 2 = 3 \cdot \frac{2}{3}
   \]

Why did we stop at \( n = 5 \)? 
\[
\frac{n-2}{n} = \frac{3}{5} \]

But 2 does not divide \( \frac{3}{5} \). 
\[
3 \cdot \frac{3}{5} < 2 < 4 \cdot \frac{3}{5}
\]

Too few 5-gons.  
Too many 5-gons.

What about \( n > 6 \)? What happens for \( n \gg 0 \)?

In general, we could ask:

For a fixed \( r \), how many of each \( n_r \)-gon \( P_n \) is needed?

Which \( r \) are possible?

For example, for \( n \) fixed only \( n = 3, 4, 6 \) is possible.
When we start to mix and match, things get more interesting. Some examples:

1) What if we mix squares and triangles? Then

\[ \frac{3 - 2}{3} + \frac{4 - 2}{4} = 2 \]

\[ \frac{1}{3} + \frac{1}{2} = 2 \]

A combination that works: \( k = 3, l = 2 \) (are there others?)

There are two possible assemblies!

2) Most interesting is when 3 different \( n \)-gons interact. What triples \( \{n_1, n_2, n_3\} \) are possible?

A surprising combination involves a \( 42 \)-gon:

\[ \frac{n_1 - 2}{n_1} + \frac{n_2 - 2}{n_2} + \frac{40}{42} = 2 \]

\[ 42 \left( n_2 (n_1 - 2) + n_1 (n_2 - 2) \right) + 40n_1n_2 = 2 \cdot 42n_1n_2 \]

\[ \frac{1}{3} + \frac{5}{7} + \frac{40}{42} = 2 \]

\[ n_1 = 3, n_2 = 7 \]

Try this! (Solution next page)
How to deduce $n_1 = 3, n_2 = 7$...

\[ 42(n_1(n_2-2) + n_1(n_2-2)) + 40n_1n_2 = 2 \cdot 42 n_1n_2 \]

\[ 2 \cdot 42 n_1n_2 - 2 \cdot 42 n_2 = 2 \cdot 42 n_1 + 40n_1n_2 = 2 \cdot 42 n_1n_2 \]

\[ 10n_1n_2 = 21(n_1 + n_2) \]

Note that $10 = 5 \cdot 2$ and $21 = 7 \cdot 3$. So

1. $n_1n_2$ should be (a multiple of) 21
2. $n_1 + n_2$ should be (a multiple of) 10

(Think about prime factors; the common multiple divides out.)

So $n_1n_2 = 21$ and $n_1 + n_2 = 10$ gives $n_1 = 3, n_2 = 7$. 