LECTURE # 14

CONSIDER $a,b,c \in \mathbb{Z}_{>2}$ AND AN a-GON
b-GON
c-GON

LAST TIME WE SAW THAT IN ORDER FOR THESE TO MEET AT A VERTEX WE NEED TO HAVE

$$\frac{a-2}{a} + \frac{b-2}{b} + \frac{c-2}{c} = 2$$

$$(a-2)bc + a(b-2)c + ab(c-2) = 2abc$$

$$3abc - 2(bc+ac+ab) = 2abc$$

$$abc = 2(bc+ac+ab)$$

THAT IS $\frac{abc}{2} = bc+ac+ab$

$$\frac{1}{2} = \frac{bc+ac+ab}{abc} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

$$\boxed{\frac{a-2}{a} + \frac{b-2}{b} + \frac{c-2}{c} = 2} \iff \boxed{\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{2}}$$

RMK

$$1 - \frac{2}{a} + 1 - \frac{2}{b} + 1 - \frac{2}{c} = 2$$

$$3 - 2\underbrace{\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)}_{\frac{1}{2}} = 2$$

REPLACING 2 BY x

$$3 - 2x^{-1} = x$$

$$3x - 2 = x^2$$

$$x^2 - 3x + 2 = 0$$

HAS ROOTS 2 AND 1.

<u>EXP</u> LET'S CHECK $\{3, 7, 42\}$ FROM LAST TIME

$$\frac{1}{3} + \frac{1}{7} + \frac{1}{\underbrace{42}_{7\cdot 3\cdot 2}} = \frac{14 + 6 + 1}{42} = \frac{21}{42} = \frac{1}{2}$$

IS THERE A SYSTEMATIC WAY TO FIND ALL ADMISIBLE $\{a, b, c\}$?

LET'S ASSUME $a \leq b \leq c$ SO THAT $\frac{1}{a} \geq \frac{1}{b} \geq \frac{1}{c}$.

THIS TELLS US THAT $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{3}{a}$

THAT IS $\frac{1}{2} \leq \frac{3}{a}$ OR $\frac{1}{6} \leq \frac{1}{a}$ OR $a \leq 6$.

> SO THE <u>SMALLEST</u> POLYGON OF THE THREE HAS <u>AT MOST</u> 6 SIDES

<u>CASE 1: $a = 6$</u> SO THAT $\frac{1}{6} + \frac{1}{b} + \frac{1}{c} = \frac{1}{2}$

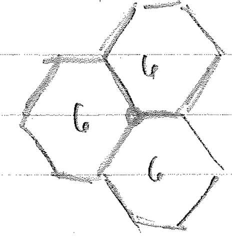
REMEMBER: $6 \leq b \leq c$ $\qquad \frac{1}{b} + \frac{1}{c} = \frac{2}{6}$

$b = c = 6$ WORKS (AS WE KNOW) $\qquad \frac{1}{b} + \frac{1}{c} = \frac{1}{3}$

OTHER POSSIBILITIES? IF $b = 7$ AND $c \geq 7$ THEN

$\frac{1}{3} > \frac{2}{7} = \frac{1}{7} + \frac{1}{7} \geq \frac{1}{b} + \frac{1}{c}$ ... SO NO!

<u>RMK</u> THIS IS THE MAXIMAL a; IT'S THE MOST CONSTRAINED



CASE 2: $a=5$. SO THAT $\frac{1}{5}+\frac{1}{b}+\frac{1}{c}=\frac{1}{2}$

$$\frac{1}{b}+\frac{1}{c}=\frac{5}{10}-\frac{2}{10}$$

$$\frac{1}{b}+\frac{1}{c}=\frac{3}{10}$$

$$\frac{3}{10}=\frac{2}{10}+\frac{1}{10}=\underbrace{\frac{1}{5}}_{\frac{1}{b}}+\underbrace{\frac{1}{10}}_{\frac{1}{c}}$$

SO $\{a,b,c\}=\{5,5,10\}$

WHAT ABOUT $\{5,6,c\}$? $\frac{1}{5}+\frac{1}{6}+\frac{1}{c}=\frac{1}{2}$

$$\frac{1}{c}=\frac{15-6-5}{30}$$

$\frac{1}{c}=\frac{4}{30}=\frac{2}{15}$ DOES NOT WORK.

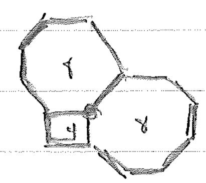
IN GENERAL, WHEN $c \geq b \geq 7$ WE GET

$$\underbrace{\frac{1}{b}+\frac{1}{c}}_{\text{WHAT WE HAVE}} \leq \frac{1}{7}+\frac{1}{7}=\frac{2}{7}<\frac{1}{2}-\frac{1}{5}=\underbrace{\frac{3}{10}}_{\text{WHAT WE NEED.}}$$

<u>CASE 3: $a=4$</u> SO THAT $\frac{1}{4}+\frac{1}{b}+\frac{1}{c}=\frac{1}{2}$ AND $c \geq b \geq 4$.

NOW $b=c=8$ WORKS SINCE $\frac{1}{8}+\frac{1}{8}=\frac{1}{4}$; $\boxed{\{4,8,8\}}$.

WE'VE SEEN THIS IN A TILING:



OTHER POSSIBILITIES?

$b=4$: $\frac{1}{4}+\frac{1}{4}+\frac{1}{c}=\frac{1}{2} \Rightarrow \frac{1}{c}=0$ $\{4,4,X\}$. NO

$b=5$: $\frac{1}{4}+\frac{1}{5}+\frac{1}{c}=\frac{1}{2} \Rightarrow \frac{1}{c}=\frac{10-4-5}{20}=\frac{1}{20}$ $\boxed{\{4,5,20\}}$ YES

$b=6$: $\frac{1}{4}+\frac{1}{6}+\frac{1}{c}=\frac{1}{2} \Rightarrow \frac{1}{c}=\frac{6-3-2}{12}=\frac{1}{12}$ $\boxed{\{4,6,12\}}$ YES

$b=7$: $\frac{1}{4}+\frac{1}{7}+\frac{1}{c}=\frac{1}{2} \Rightarrow \frac{1}{c}=\frac{14-7-4}{28}=\frac{3}{28}$ $\{4,7,X\}$ NO

$b=8$: WE DID THIS ONE!

$b=9$: $\frac{1}{4}+\frac{1}{9}+\frac{1}{c}=\frac{1}{2} \Rightarrow \frac{1}{c}=\frac{18-9-4}{2}=\frac{5}{2}$ $\{4,9,X\}$ NO.

SHOULD WE GIVE UP? IF $b=10$ AND $c \geq 10$ THEN

$$\frac{1}{10}+\frac{1}{c} \leq \frac{1}{10}+\frac{1}{10}=\frac{1}{5}<\frac{1}{4}$$

SO $\frac{1}{4}+\frac{1}{10}+\frac{1}{c}<\frac{1}{2}$ FOR ALL POSSIBLE c!

AMAZINGLY, 2 OF THESE 3 "SPECIES" ARISE IN A TILING: