Lecture #15

We saw that \( \frac{1}{3} + \frac{1}{7} + \frac{1}{42} = \frac{1}{2} \) corresponds to the vertex.

\[
\frac{1}{3} \pi + \frac{5}{7} \pi + \frac{40}{42} \pi = 2 \pi
\]
\[
\Rightarrow \frac{14 + 30 + 40}{42} \pi = \frac{73}{3} \pi
\]

Try to extend this to a "uniform" tiling, where every vertex is of this type (and all edges have same length).

**Angle Sum Works at Vertex y**

What is \( \Theta \)? In order to get a \( \{3, 7, 42\} \) vertex, it has to be

\[
\frac{40}{42} \pi = \frac{20}{21} \pi
\]

That is, we require a \( \{3, 7, 42\} \) vertex, but also

\[
\frac{1}{3} \pi + \frac{5}{7} \pi + \frac{40}{42} \pi = 2 \pi \quad \Leftrightarrow \quad \frac{1}{3} \pi + \frac{5}{7} \pi + \frac{5}{7} \pi + \frac{40}{42} \pi > 2 \pi
\]

\[
\Theta = 2 \pi - \frac{1}{3} \pi - \frac{10}{7} \pi = \frac{42 - 7 - 30}{21} \pi = \frac{5}{21} \pi \neq \frac{1}{3} \pi + \frac{5}{7} \pi + \frac{20}{21} \pi
\]
So while there are 19 vertex species, and 21 vertex types, there are only 11 uniform tilings.

exp \{3,3,3,4,4\} vertex species #3 from G6s.

Two vertex types:

\[ 3.3.3.4.4 \]  \[ 3.3.4.3.4 \]

Tilings exist in both cases (this part is constructive):

G6s notation: \( (3^3.4^2) \) \( (3^2.4.3.4) \)

These 11 tilings — see G6s 2.1.3 on page 59 — are "uniform" in the following sense:

For any two vertices \( x, y \in T \), there is a symmetry \( g \in S(\mathcal{T}) \) so that \( g(x) = y \).

Terminology: every vertex is in the same "transitivity class."
REMARK THAT WE SAW A MONOCHROMATIC TILING IS ISOMETRIC
IF \( T_1, T_2 \in G \) CAN BE RELATED BY \( gT_1 = T_2 \) FOR
\( g \in S(G) \).

NOTE THAT SUB-OBJECTS IN A GIVEN \( G \) MAY OR MAY
NOT BE RELATED BY A SYMMETRY. THIS CAN BE
PHRASED AS AN EQUIVALENCE RELATION:

GIVEN \( T_1, T_2 \in G \) DEFINE \( T_1 \sim T_2 \) WHEN THERE
IS A \( g \in S(G) \) SO THAT \( g(T_1) = T_2 \).

REFLEXIVITY: \( T \sim T \) FOR ALL \( T \in G \)

SYMMETRY: \( T_1 \sim T_2 \iff T_2 \sim T_1 \)

TRANSITIVITY: \( T_1 \sim T_2 \) AND \( T_2 \sim T_3 \) IMPLIES \( T_1 \sim T_3 \).

AN "EQUIVALENCE CLASS" IS THE SET OF ALL
SUB-OBJECTS (TILES, IN THIS CASE) RELATED BY
THIS EQUIVALENCE. SO

A \( k \)-ISOMETRICAL TILING HAS EXACTLY \( k \) EQUIVALENCE
CLASSES OF TILES.

A "UNIFORM TILING" (BY REGULAR POLYGONS) HAS
ONE EQUIVALENCE OF VERTEX (i.e., THE VERTEX
TYPE).

INTERESTING GENERALIZATION: \( k \)-UNIFORM SO THAT
THERE ARE \( k \)-EQUIVALENCE CLASSES OF VERTICES.

THIS ALLOWS US TO PICK \( k \) VERTEX TYPES FROM OUR
POSSIBLE 21, AND TRY TO PIECE THEM INTO A TILING.
These are distinct 2-uniform edge to edge tilings by regular polygons.