LECTURE # 18

HOUSE KEEPING...

WE HAVE SEEN THAT TILINGS $\mathcal{T}_1$ AND $\mathcal{T}_2$ ARE CONGRUENT IF THERE IS AN AFFINE TRANSFORMATION $g: \mathbb{R}^2 \to \mathbb{R}^2$ SUCH THAT $g\mathcal{T}_1$ AGREES (ON TILES, EDGES, VERTICES...) WITH $\mathcal{T}_2$. *WHERE g HAS ORTHOGONAL LINEAR COMPONENT*

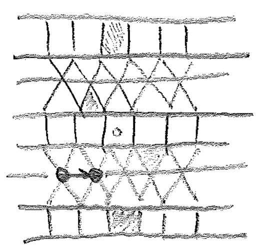
EXAMPLES OF POSSIBLE g'S INCLUDE TRANSLATIONS AND ROTATIONS.

IMPORTANT: THESE ARE FUNCTIONS ON $\mathbb{R}^2$! $\boxed{g(x)-g(y) = f(x-y)}$.

THE SYMMETRIES OF A GIVEN TILING $\mathcal{T}$ IS THE SET OF FUNCTIONS:

$$S(\mathcal{T}) = \{ g: \mathbb{R}^2 \to \mathbb{R}^2 \mid g \text{ CARRIES } \mathcal{T} \text{ TO ITSELF} \}.$$

EXAMPLE:



* TRANSLATION $t(x,y) = (x+1, y)$
* ROTATION ABOUT o BY π.

IMPORTANT: SYMMETRIES COMPOSE — AS FUNCTIONS — TO GIVE NEW SYMMETRIES. A RECURRING THEME HAS BEEN "COLLECT SUB-OBJECTS OF $\mathcal{T}$ UP TO SYMMETRIES IN $S(\mathcal{T})$."

MOST RECENTLY, THESE SUB-OBJECTS HAVE BEEN VERTICES (IN k-UNIFORM TILINGS). SO LET'S STICK WITH VERTICES FOR CONCRETENESS.

THE *RIGHT* WAY TO DO THIS IS BY DECLARING AN EQUIVALENCE:

$$v \sim v' \iff g(v) = v' \text{ FOR SOME } g \in S(\mathcal{J}).$$

"v IS EQUIVALENT TO v'"

DEFINITION OF DESIRED EQUIVALENCE ...

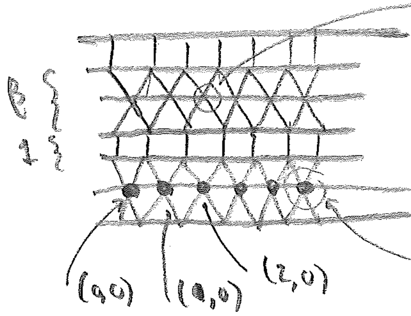
... UP TO SYMMETRIES OF $\mathcal{J}$.

NOW WE HAVE SETS OF VERTICES THAT DIFFER BY SOME SYMMETRY, CALLED "EQUIVALENCE CLASSES":

$$[v] = \{ \text{VERTICES } v' \mid v' \sim v \}.$$

NOTE THAT TWO $v', v'' \in [v]$ ARE DIFFERENT CHOICES OF REPRESENTATIVE FOR THE SAME EQUIVALENCE CLASS. IN PARTICULAR, $v \in [v]$ IS ALWAYS A REPRESENTATIVE!

RETURNING TO OUR EXAMPLE FROM LAST TIME:



$v' = (m, \beta+1)$

AND $(l, \beta+1) \in [v'] \; \forall l \in \mathbb{Z}$.

$v = (n, 0)$ SO: $(l, 0) \in [v] \; \forall l \in \mathbb{Z}$

THESE ASSERTIONS MAKE USE OF $t \in S(\mathcal{J})$ WHERE

$$t(x,y) = (x+1, y).$$

QUESTION: WHAT WAS v'' FROM LAST TIME? IT WAS THE INTERMEDIATE REP $v'' = (m, 0) \in [v]$ SO THAT

1. $v'' \sim v$
2. $t_\beta(m,0) = (m, \beta+1) = v'$ AND $v'' \sim v'$

THIS PROVES THAT ALL TR-VERTICES ARE IN THE SAME EQUIVALENCE CLASS, WHICH IS WHAT WE WANTED.

THINGS YOU CAN DO THAT WILL HELP:

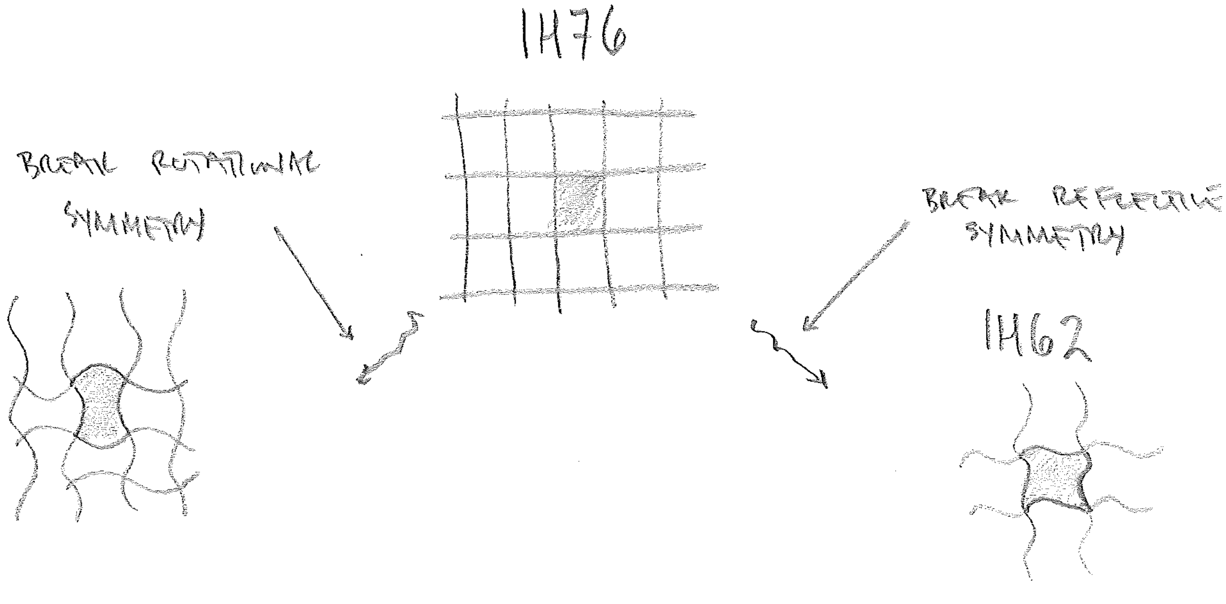
1. TRY THE SAME CAREFUL SETUP TO SHOW ALL SQ-VERTICES ARE EQUIVALENT
2. REVIEW ISOHEDRALITY AND k-ISOHEDRALITY IN TERMS OF EQUIVALENCE OF TILES

NEXT UP: TILINGS BY GENERAL POLYGONS.

SOME COMMENTS ON MATERIAL WE WILL PASS OVER.

[A] CLASSIFICATION OF ISOHEDRAL TILINGS. (Ch 6).

FROM G&S FIGURE 6.2.4



WE CONSTRUCTED SUCH THINGS BY CUT AND PASTE.

ALTERNATIVELY, THESE "DEFORMATIONS" CAN BE ACHIEVED BY CONTINUOUS FUNCTIONS

$$f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

(THAT ARE BIJECTIVE) IN THE EXAMPLES ABOVE, THESE f ARE NOT LINEAR!

SUCH f ARE CALLED "HOMEOMORPHISMS", WHICH CAN BE THOUGHT OF AS DEFORMATIONS THAT BEND OR TWIST $\mathbb{R}^2$ BUT DO NOT RIP OR TEAR IT.

[B] TOPOLOGY OF TILINGS. (Ch 4)

WHEN WE RESTRICTED TO DISKS FOR TILES, THE TERM "TOPOLOGY" POPPED OUT — SO WE'VE SEEN HOMEOMORPHISMS ALREADY.

WE WON'T DELVE INTO THIS, BUT IT IS WORTH NOTING THAT EQUALITY OF TILINGS (AS WE'VE DEFINED) AND TOPOLOGICAL EQUIVALENCE OF TILINGS IS VERY DIFFERENT — A3E3 SEES THIS COME UP.

LOOSELY SPEAKING: TOPOLOGY STUDIES OBJECTS UP TO HOMEOMORPHISM, i.e., UP TO CONTINUOUS DEFORMATION.

NOTE: "OBJECTS" HERE SHOULD BE BROADLY INTERPRETED.