LAST TIME: HOMEOMORPHISM AND CONTINUITY.

A HOMEOMORPHISM \( \varphi : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) is a continuous bijection (with continuous inverse).

RMK: IF \( A \) is a \( 2 \times 2 \) matrix it is invertible if \( \det(A) \neq 0 \) and the linear transformations from \( A \) and \( A^{-1} \).

Now any affine \( \varphi : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) with linear part \( A \) is a homeomorphism.

For \( \varphi \) to be compatible with \( S(\varphi_1) \) and \( S(\varphi_2) \) we require:

For any \( g \in S(\varphi_1) \) there exists a \( g_2 \in S(\varphi_2) \) such that

\[
\varphi_1 \rightarrow \varphi_2 \\
g_1 \rightarrow \varphi \rightarrow g_2 = \varphi \circ g_1 \circ \varphi^{-1}
\]

RMK: For \( x \in \mathbb{R}^2 \) we have \( g_2(x) = (\varphi \circ g_1 \circ \varphi^{-1})(x) = \varphi(g_1(\varphi(x))) \)

or

\[
(g_2 \circ \varphi)(x) = (\varphi \circ g_1)(x)
\]

\( \psi \circ g_1 \circ \psi^{-1} \in S(\varphi_2) \) so if \( \varphi \) compatible \( \psi \) exists, \( S(\varphi_1) \) and \( S(\varphi_2) \) agree.

We have made use of this fact: if a planar operation \( \psi \) is a symmetry, then \( \varphi_1 \) and \( \varphi_2 \) are different.

IMPORTANT: "BEING EQUAL" AND "BEING OF THE SAME TYPE" ARE NOT THE SAME THING!
REAP FROM LAST TIME.

Corners are elements of a tile \( T \).

\[ \begin{array}{c}
\text{PT} \\
\text{of } T
\end{array} \quad \begin{array}{c}
\text{ Graph of some } f(x) \\
\text{ which is continuous and has a }
\text{derivative at } a
\end{array} \]

\[ f \text{ does not have a }
\text{derivative at } a \]

\( T \) is a continuous non-intersecting derivative here; this
came in the plane; it has
a finite number of corners \( \{ c_i \}_{i=1}^N \).

Sides of \( T \) := components of \( \partial T \) \( \{ c_i \} \) when \( T \) is
a polygon; sides
are line segments.

Vertices are elements of a thing \( J \).

Consider \( T_1, T_2, \ldots, T_n \in J \). We have either:

\[ \bigcap_{i=1}^n T_i \neq \phi \quad \text{or} \quad \bigcap_{i=1}^n T_i \subset \partial T_i \quad i = 1, 2, \ldots, n. \]

So intersections consist of points and arcs.

\[ \text{ "vertices" } \quad \text{ "edges"} \]

\( \text{ EXP } \)

\[ \begin{array}{c}
T_1 \\
T_2 \\
T_3
\end{array} \quad \begin{array}{c}
\text{ edge} \\
\text{ vertex}
\end{array} \]

Remark: not all corners are vertices; a vertex
is the corner of some tile.
In a normal tiling, to fill this in, we need some non-regular tiling.

Homothety.

Def \( T_1 \) and \( T_2 \) are "homothetic" or "homothety equivalent" if there is a compatible homeomorphism \( \Psi : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \)

such that \( \Psi(0,0) = T_2 \).

\[ \text{Write } T_1 \leftrightarrow T_2 \]

Exp \( \Psi : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \)

\[ (x,y) \rightarrow (x, y + \alpha \sin(\pi x)) \]. For \( \alpha > 0 \).

What does this do to the x axis?

\[ \Psi(x,0) = (x, \alpha \sin(\pi x)) \]

This changes:

\[ \begin{array}{c|c}
\hline
0 & 1 \\
\hline
\end{array} \]

\[ \begin{array}{c|c}
\hline
0 & 1 \\
\hline
\end{array} \]

This becomes:
Now if \( T_\alpha \) is the tiling we get from lines
\[ \{ x = n \} \cup \mathbb{Z} \quad \text{and} \quad \{ y = \alpha \sin(\pi x) + n \} \cup \mathbb{Z} \]

then

1. \( T_0 \) is not homothety equivalent to \( T_\alpha \) for any \( \alpha > 0 \).

2. \( T_\alpha \) is homothety equivalent to \( T_\alpha' \) for any \( \alpha > \alpha' > 0 \).

Hence all corners are vertices in this example.

Claim: \( T_\alpha \) homothetic to \( T_\alpha \) for all \( \alpha \). Hence
\( T_\alpha \) homothetic to \( T_\alpha' \) for any \( \alpha, \alpha' \in \mathbb{R}^+ \).

Good time to remind yourself what \( \Rightarrow \) means for tilings.
"Homotopy is an equivalence relation on tilings:

\[ T_1 \xrightarrow{\Psi} T_2 \iff \Psi \text{ is a compatible homeomorphism.} \]

Terminology: \( \Psi \) is the homotopy from \( T_1 \) to \( T_2 \).

Since \( \Psi \) is a bijection, we always have \( T_2 \xrightarrow{\Psi^{-1}} T_1 \) as well.

(This is part of the check it's an equivalence relation.)

Homotopy is weaker than equality.

\[ \text{Homomorphism...} \quad \text{Similarity...} \]

(As if tilings are equal they are homotopic, but...)

Pop a homotopy takes vertices of \( T_1 \) to vertices of \( T_2 \).

Proof: Let \( v_i \) vertex of \( T_1 \); \( i \equiv 1, 2 \).

Given \( \Psi: T_1 \rightarrow T_2 \) and \( g \in S(T_2) \)

compatibility allows us to construct

\[ \Psi^* g, \Psi \in S(T) \]

Now \( g(v_0) \) must be a vertex of \( T_2 \):

\[(\Psi^* g, \Psi)(v_1) \text{ must be a vertex of } T_1. \]

The only way for this to happen is with \( \Psi(v_1) = v_2 \).
\[
Y(x, y) = (x, y + \alpha \sin(\pi x)) \text{ for } \alpha > 0.
\]

This fixes the lines \( \{ x = n \} \text{ for } n \in \mathbb{Z} \).

It carries lines \( \{ y = n \} \) to curves \( \{ y = \alpha \sin(\pi x) + n \} \).

* Vertices go to vertices, as claimed.
  *(and all corners are vertices)*.

\[
T_{\alpha'} = \text{new times}
\]

Call the result \( T_{\alpha'} \); note \( T_{0} = T_{0} \).

Claim: \( T_{\alpha} \) is homotopic to \( T_{0} \) for all \( \alpha \);
\( T_{\alpha} \) is homotopic to \( T_{\alpha'} \) for any \( \alpha > \alpha' > 0 \).
WE NEED TO BUILD \[ J' : J_0 \rightarrow J'_0 \]

THIS \( J' \) CAN'T BE LINEAR (WHY?)

BUT IT IS LINEAR IN VERTICAL STRIPS:

CONSIDER \([n, n + \frac{1}{2}] \times \mathbb{R}\) FOR \(n \in \mathbb{Z}\).

\[
\begin{array}{c|c|c}
\hline
m & m + 1 & m + 2 \\
\hline
n & n + \frac{1}{2} & n + 1 \\
\hline
\end{array}
\]

TYPICAL TILE

NOTE: THIS IS A HOMEOMORPHISM ON \([n, n + \frac{1}{2}] \times \mathbb{R}\).

NOW CONSIDER \([n + \frac{1}{2}, n + 1] \times \mathbb{R}\) FOR \(n \in \mathbb{Z}\).

\[
\begin{array}{c|c|c}
\hline
m & m + 1 & m + 2 \\
\hline
n + \frac{1}{2} & n + 1 \\
\hline
\end{array}
\]

THESE SHEARS AGREE ON \([n + \frac{1}{2}, n + 1] \times \mathbb{R}\), SO WE GET A HOMEOMORPHISM ON \([n, n + 1] \times \mathbb{R}\) THAT IS THE IDENTITY ON \(\mathbb{Z} \times \mathbb{R}\).

NEXT TIME: POLYGONAL ISOHEXRAL TYPE.