LAST TIME: HOMEOMORPHISM AND COMPATIBILITY.

A HOMEOMORPHISM $\varphi: \mathbb{R}^2 \to \mathbb{R}^2$ IS A CONTINUOUS BIJECTION (WITH CONTINUOUS INVERSE).

RMK · IF A IS A 2×2 MATRIX IT IS INVERTIBLE IF $\det(A) \neq 0$ AND THE LINEAR TRANSFORMATIONS FROM A AND A^{-1}.

· NOW ANY AFFINE $g: \mathbb{R}^2 \to \mathbb{R}^2$ WITH LINEAR PART A IS A HOMEOMORPHISM.

FOR φ TO BE COMPATIBLE WITH $S(\mathcal{T}_1)$ AND $S(\mathcal{T}_2)$ WE REQUIRE

FOR ANY $g_1 \in S(\mathcal{T}_1)$ THERE EXISTS A $g_2 \in S(\mathcal{T}_2)$ SUCH THAT

$$\begin{tikzcd} \mathcal{T}_1 \arrow[r, "\varphi"] \arrow[d, "g_1"'] & \mathcal{T}_2 \arrow[d, dashed] \\ \mathcal{T}_1 \arrow[r, "\varphi"] & \mathcal{T}_2 \end{tikzcd} \qquad \boxed{g_2 = \varphi \circ g_1 \circ \varphi^{-1}}$$

RMK · FOR $x \in \mathbb{R}^2$ WE HAVE $g_2(x) = (\varphi \circ g_1 \circ \varphi^{-1})(x) = \varphi(g(\varphi^{-1}(x)))$

OR

$$(g_2 \circ \varphi)(x) = (\varphi \circ g_1)(x)$$

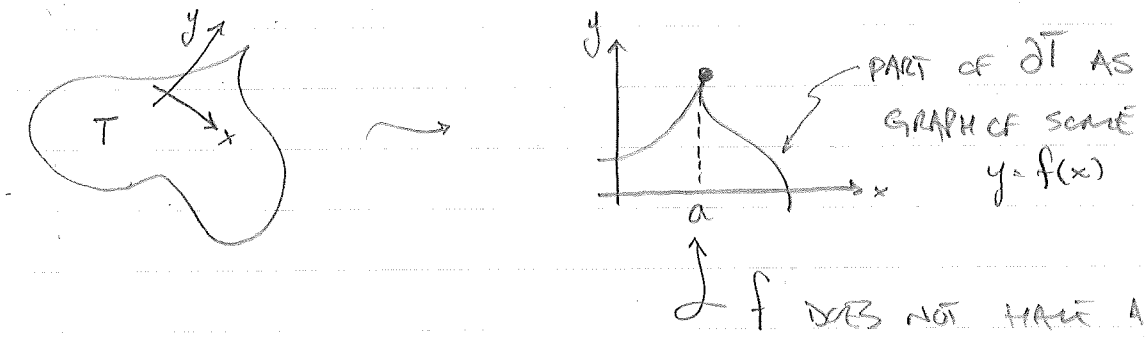
· $\varphi \circ g_1 \circ \varphi^{-1} \in S(\mathcal{T}_2)$ SO IF A COMPATIBLE φ EXISTS, $S(\mathcal{T}_1)$ AND $S(\mathcal{T}_2)$ AGREE.

WE HAVE MADE USE OF THIS FACT: IF A PLANAR OPERATION BREAKS A SYMMETRY, THEN $\mathcal{T}_1$ AND $\mathcal{T}_2$ ARE DIFFERENT.

IMPORTANT: "BEING EQUAL" AND "BEING OF THE SAME TYPE" ARE NOT THE SAME THING!

RECAP FROM LAST TIME.

CORNERS ARE ELEMENTS OF A TILE T.



∂T IS A CONTINUOUS NON-INTERSECTING CURVE IN THE PLANE; IT HAS A FINITE NUMBER OF CORNERS $\{c_i\}_{i=1}^{N}$

SIDES OF T := COMPONENTS OF $\partial T \setminus \{c_i\}$ || WHEN T IS A POLYGON, SIDES ARE LINE SEGMENTS.

VERTICES ARE ELEMENTS OF A TILING $\mathcal{T}$.

CONSIDER $T_1, T_2, \dots, T_n \in \mathcal{T}$. WE HAVE EITHER:

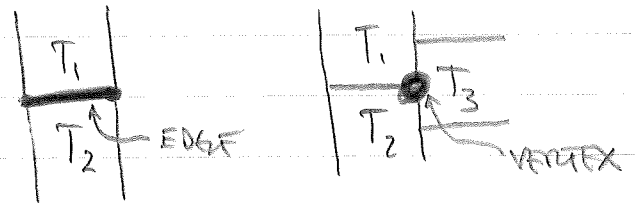
$$\bigcap_{i=1}^{n} T_i \neq \emptyset \quad \text{OR} \quad \bigcap_{i=1}^{n} T_i \subset \partial T_k \quad i = 1, 2, \dots, n.$$

NO OVERLAPS!!!

SO INTERSECTIONS CONSIST OF POINTS AND ARCS.

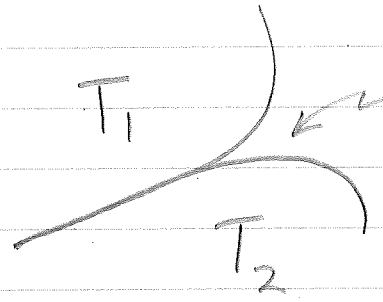
"VERTICES" "EDGES"

EXP

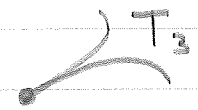


RMK NOT ALL CORNERS ARE VERTICES; A VERTEX IS THE CORNER OF SOME TILE.

EXP



IN A NORMAL TILING, TO FILL THIS IN, WE NEED SOME



NO DEFLATION HERE!

HOMOTHETY.

DEF $\mathcal{T}_1$ AND $\mathcal{T}_2$ ARE "HOMOTHETIC" OR "HOMOTHETY EQUIVALENT" IF THERE IS A COMPATIBLE HOMEOMORPHISM

$$\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

SUCH THAT $\varphi(\mathcal{T}_1) = \mathcal{T}_2$ $\quad$ [WRITE $\mathcal{T}_1 \xrightarrow{\varphi} \mathcal{T}_2$].

EXP

$$\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(x,y) \longmapsto (x, y + \alpha \sin(\pi x)). \quad \text{FOR } \alpha > 0.$$

WHAT DOES THIS DO TO THE X AXIS?

$$\varphi(x, 0) = (x, \alpha \sin(\pi x))$$

THIS CARRIES: $\qquad$ TO:

NOW IF $\mathcal{T}_\alpha$ IS THE TILING WE GET FROM LINES

$$\{x = n\}_{n \in \mathbb{Z}} \quad \text{AND} \quad \{y = \alpha \sin(\pi x) + n\}_{n \in \mathbb{Z}}$$

THEN

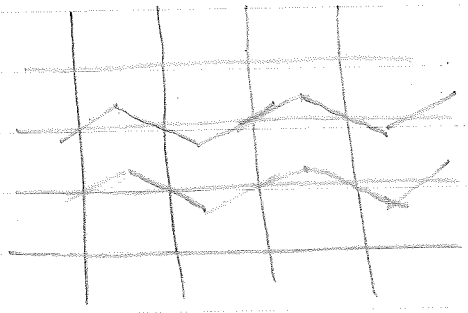
1. $\mathcal{T}_0$ <u>NOT</u> HOMOTHETY EQUIVALENT TO $\mathcal{T}_\alpha$ FOR ANY $\alpha > 0$.

2. $\mathcal{T}_\alpha$ HOMOTHETY EQUIVALENT TO $\mathcal{T}_{\alpha'}$ FOR ANY $\alpha > \alpha' > 0$.

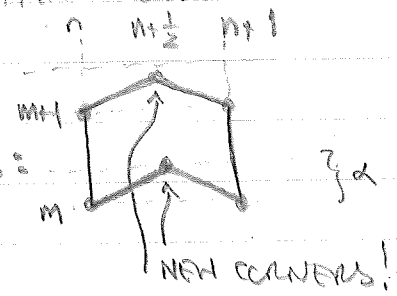
<u>EXC</u> VERIFY (1) AND (2).

<u>RMK</u> ALL CORNERS ARE VERTICES IN THIS EXAMPLE.

<u>EXP</u>



NEW TILES:



CALL THE RESULT $\mathcal{T}'_\alpha$; NOTE THAT $\mathcal{T}_0 = \mathcal{T}'_0$

GOOD TIME TO REMIND YOURSELF WHAT $=$ MEANS FOR TILINGS.

CLAIM: $\mathcal{T}'_\alpha$ ~~HOMOTHETIC~~ HOMOTHETIC TO $\mathcal{T}'_0$ FOR ALL α. HENCE

$\mathcal{T}'_\alpha$ HOMOTHETIC TO $\mathcal{T}_{\alpha'}$

FOR <u>ANY</u> $\alpha, \alpha' \in \mathbb{R}_{>0}$.

"HOMOTHETY" IS AN EQUIVALENCE RELATION ON TILINGS:

$$\mathcal{T}_1 \xrightarrow{\varphi} \mathcal{T}_2 \iff \varphi \text{ IS A COMPATIBLE HOMEOMORPHISM.}$$

TERMINOLOGY: φ IS THE HOMOTHETY FROM $\mathcal{T}_1$ TO $\mathcal{T}_2$.

REMARK SINCE φ IS A BIJECTION, WE ALWAYS HAVE $\mathcal{T}_2 \xrightarrow{\varphi^{-1}} \mathcal{T}_1$ AS WELL.

(THIS IS PART OF THE CHECK IT'S AN EQUIVALENCE RELATION).

HOMOTHETY IS WEAKER THAN EQUALITY

HOMEOMORPHISM... SIMILARITY...

(AN IF TILINGS ARE EQUAL THEY ARE HOMOTHETIC, BUT...)

PROP A HOMOTHETY TAKES VERTICES OF $\mathcal{T}_1$ TO VERTICES OF $\mathcal{T}_2$.

PROOF LET v_i VERTEX OF $\mathcal{T}_i$; $i = 1, 2$.

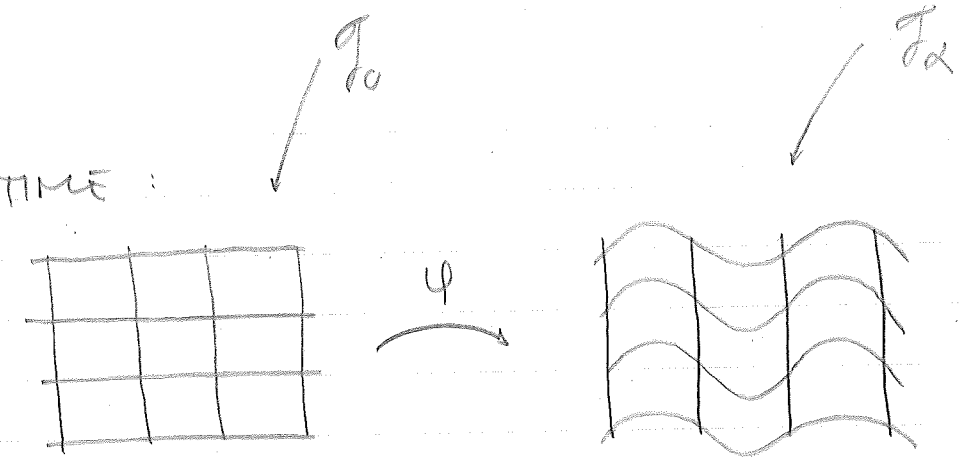
GIVEN $\varphi : \mathcal{T}_1 \longrightarrow \mathcal{T}_2$ AND $g \in S(\mathcal{T}_2)$
COMPATIBILITY ALLOWS US TO CONSTRUCT

$$\varphi^{-1} \circ g \circ \varphi \in S(\mathcal{T}_1)$$

NOW $g(v_2)$ MUST BE A VERTEX OF $\mathcal{T}_2$;
$(\varphi^{-1} \circ g \circ \varphi)(v_1)$ MUST BE A VERTEX OF $\mathcal{T}_1$.

THE ONLY WAY FOR THIS TO HAPPEN IS WITH $\varphi(v_1) = v_2$ ∎.

LAST TIME:



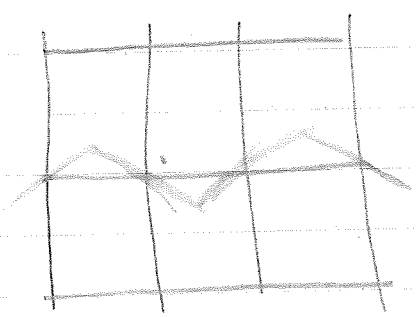
$\varphi(x,y) = (x,\ y + \alpha\sin(\pi x))$ FOR $\alpha > 0$.

THIS · <u>FIXES</u> THE LINES $\{x = n\}_{n\in\mathbb{Z}}$.

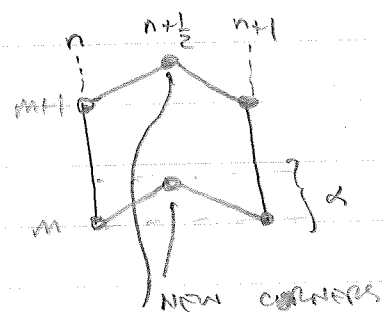
· CARRIES LINES $\{y = n\}$ TO CURVES $\{y = \alpha\sin(\pi x) + n\}$

* VERTICES GO TO VERTICES, AS CLAIMED*.
(AND ALL CORNERS ARE VERTICES).

<u>EXP</u>



NEW TILES



CALL THE RESULT $\mathcal{T}_\alpha'$; NOTE $\mathcal{T}_0 = \mathcal{T}_0'$

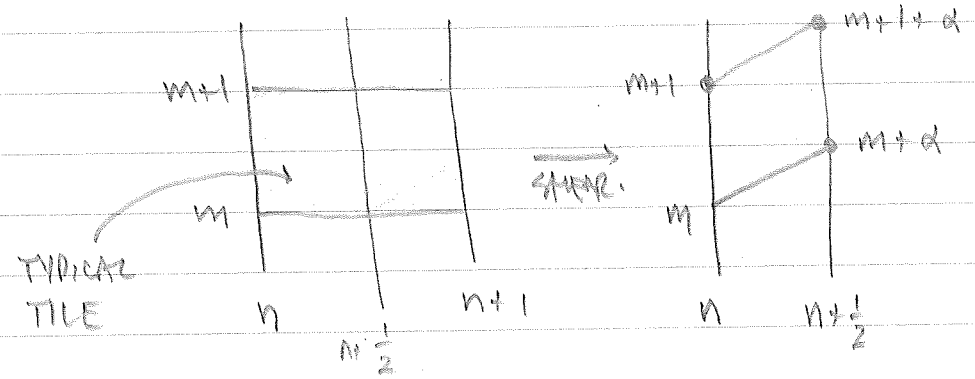
CLAIM: $\mathcal{T}_\alpha'$ HOMOTOPIC TO $\mathcal{T}_0'$ FOR ALL α;
$\mathcal{T}_\alpha'$ HOMOTHETIC TO $\mathcal{T}_{\alpha'}'$ FOR ANY $\alpha > \alpha' > 0$.

WE NEED TO BUILD $\mathcal{T}_\theta' \xrightarrow{\varphi'} \mathcal{T}_\alpha'$

THIS φ' <u>CAN'T</u> BE LINEAR (WHY?)

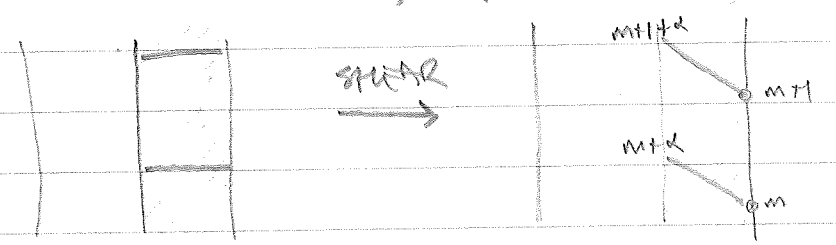
BUT IT <u>IS</u> LINEAR IN VERTICAL STRIPS:

CONSIDER $[n, n+\frac{1}{2}] \times \mathbb{R}$ FOR $n \in \mathbb{Z}$.



NOTE: THIS IS A HOMEOMORPHISM ON $[n, n+\frac{1}{2}] \times \mathbb{R}$.

NOW CONSIDER $[n+\frac{1}{2}, n+1] \times \mathbb{R}$ FOR $n \in \mathbb{Z}$.



THESE SHEARS AGREE ON $\{n+\frac{1}{2}\} \times \mathbb{R}$, SO WE GET A HOMEOMORPHISM ON $[n, n+1] \times \mathbb{R}$ THAT IS THE IDENTITY ON $\{n\} \times \mathbb{R}$.

NEXT TIME: POLYGONAL ISOTHEDRAL TYPE.