Math 427/527: algebraic topology

Homework 1, due Friday February 11 by 5:00 pm.

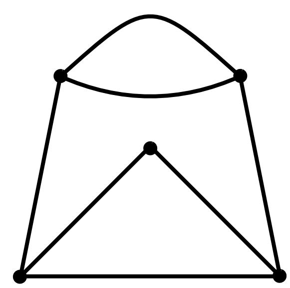
1. Fix a category $\mathcal{C}$.

(a) Prove that if $f, g \in \mathrm{ar}(\mathcal{C})$ are composable (that is, composable morphisms in $\mathcal{C}$) such that $g \circ f$ and g are isomorphisms, then f is an isomorphism as well.

(b) Prove that if $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ is a sequence of morphisms in $\mathcal{C}$ such that $g \circ f$ and $h \circ g$ are isomorphisms, then g (and therefore f and h) must be isomorphisms.

(c) Now let $\mathcal{C}$ be Top, the category of topological spaces. Give the definition of a deformation retract and, using part (ii), show that a deformation retract is a homotopy equivalence.

2. In Lecture 4 (January 21) we developed a scheme, as a means of providing motivation and generating intuition, for calculating the first homology group of a graph. For this problem, consider the following graph Γ:



(a) Calculate the homology $H_1(\Gamma; \mathbf{Z})$ according to our scheme from Lecture 4, in particular, you should give a basis in terms of cycles. Show all of your work.

(b) In part (a), you needed to choose orientations on each edge. Prove that your answer does not depend on this choice.

(c) By finding an appropriate presentation for $\pi_1(\Gamma)$, prove that

$$H_1(\Gamma; \mathbf{Z}) \cong \frac{\pi_1(\Gamma)}{\big[\pi_1(\Gamma), \pi_1(\Gamma)\big]}$$

3. Recall that $X \times_c Y = k(X \times Y)$.

(a) Give examples of CW-complexes X and Y for which $X \times Y$ is not a CW-complex. Justify your answer.

(b) If X and Y are CW-complexes, with characteristic maps Φ_α and Ψ_β, respectively, prove that $X \times_c Y$ is a CW-complex with characteristic maps $\Phi_\alpha \times \Psi_\beta$.

4. Let X be a CW-complex. The *unreduced* suspension SX of X is the quotient space $X \times I/\!\sim$ where $(x, 0) \sim (y, 0)$ for all $x, y \in X$ and $(x, 1) \sim (y, 1)$ for all $x, y \in X$. The *reduced* suspension then is just the usual the suspension: $\Sigma X = X \wedge S^1$. By factoring a choice of map $X \times I \to \Sigma X$ through SX, show that SX is homotopy equivalent to ΣX. (You may find it useful to consult Proposition 0.17 in Hatcher.)