Math 427/527: algebraic topology

Homework 4, due Wednesday April 13 by 5:00 pm.

1. For this problem X is a CW complex that is finite and connected, and $\widehat{X}$ is the infinite cyclic cover associated with a surjection $\pi_1(X)$ onto an infinite cyclic group Π. Recall that $\mathbf{k}\Pi$ denotes the group ring where $\mathbf{k}$ is a field.

(i) If t a generator for Π, show that

$$0 \to C_*(\widehat{X};\mathbf{k}) \to C_*(\widehat{X};\mathbf{k}) \to C_*(X;\mathbf{k}) \to 0$$

is a short exact sequence of complexes where the map on the left is given by $t-1$. Deduce, stating and applying the snake lemma, the associated long exact sequence on homology.

(ii) Prove that if X has the homology of a circle, then $\mathrm{ord}_{\mathbf{k}\Pi}\left(H_i(\widehat{X};\mathbf{k})\right)$ is non-zero for all i. (In particular, this shows that $H_i(\widehat{X};\mathbf{k})$ is finitely generated as a $\mathbf{k}$-vector space.)

(iii) Prove that if $H_i(\widehat{X};\mathbf{k})$ is finitely generated as a $\mathbf{k}$-vector space the the euler characteristic $\chi(X)$ vanishes.

2. For each of the following X, showing your work, determine $H_i(\widehat{X};\mathbf{k})$ as a $\mathbf{k}\Pi$-module:

(i) when X is the Klein bottle; and

(ii) when X is $S^2 \times S^1$.