/isual Mind
*l Mathematics*
*by Michele Emmer*

ific visualization, higher-dimensional geome-
three-dimensional computer modeling, com-
animation, and virtual environments are just a
the ground-breaking areas in which artists and
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35 chapters are written by mathematicians
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visual artists concerned with the mathematical
and inspirations of their works. They are
into sections covering geometry and visualiza-
omputer graphics, geometry, and art; symme-
l perspective, mathematics, and art. The chap-
e tied together by introductions to each part
richly illustrated in both color and black and

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ità ca' Foscari, Venice, Italy.

*rdo* Book


Polytopes in Six and Seven
*hical Society* **24** (1928) pp. 1–9.

on p. 126.

**26.** See Coxeter [2] p. 134.

**27.** See Coxeter [2] p. 135.

**28.** See Coxeter [2] Eq. 13.691 on p. 152.


CHAPTER 5

# On the Cohomology of Impossible Figures


*Roger Penrose*



ABSTRACT

The close relationship between certain types of impossible figure and the mathematical idea of cohomology is explained in relation to the tribar and to another type of impossible figure related to the Necker cube.


In a recent article [1], presented in honour of M. C. Escher, I hinted at a relationship between cohomology and certain types of impossible figure. It is the purpose of this note to explain this relationship more fully.

I shall be concerned with the concept of *first* cohomology group

$$H^1\ (Q, G); \qquad (1)$$

the basic meaning of this concept should emerge during the course of the discussion. Here Q is some (non-simply connected) region of the plane—which I shall take to contain the 'support' (i.e. the region of the plane where the drawing occurs) of some impossible figure—and G is a (normally Abelian) group, which I shall refer to as the *ambiguity group* of the figure. (For those readers not familiar with the mathematical concept of a group, it may be taken that G is just some set of numbers closed under multiplication and division. Thus if a and b belong to G, then so do ab and a/b.) To fix ideas, let us consider two examples. The first is the tribar, illustrated in Fig. 1. Here, Q can be taken to be, say, the region of the plane (paper) on which the tribar is actually drawn, or else some slightly larger region such as the annular region depicted in Fig. 2. In the second example, illustrated in Fig. 3, I have drawn a version of impossible figure that I introduced in my earlier article.

Consider first the tribar. We may regard the region Q as being pasted together from three smaller regions $Q_1$, $Q_2$, $Q_3$, as indicated in Fig. 4. There are overlapping pars of $Q_1$, $Q_2$, $Q_3$, which are to be pasted together.

The drawing, on each of $Q_1$, $Q_2$, $Q_3$, is a perfectly consistent rendering of a three-dimensional structure that is unambiguous in its natural interpretation—except for the essential ambiguity present in all pictures: one does not know the *distance* away from the observer's eye that the object being depicted is supposed to be situated (Fig. 5). Of


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**Fig. 1. An impossible figure, the tribar, drawn in perspective.**

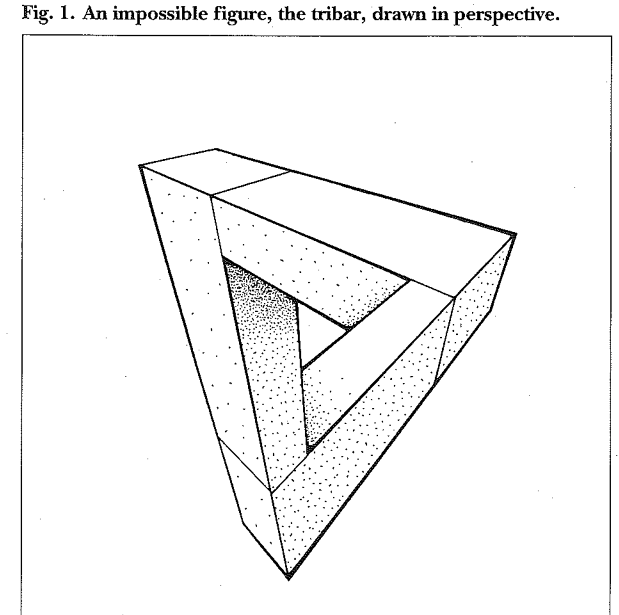
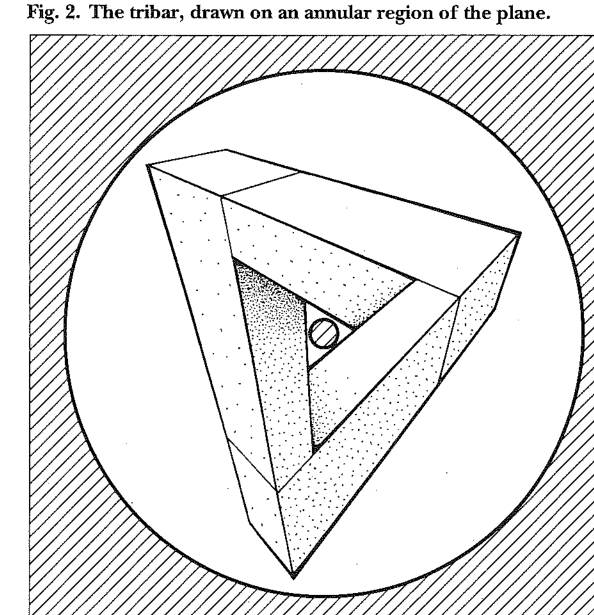


**Fig. 2. The tribar, drawn on an annular region of the plane.**



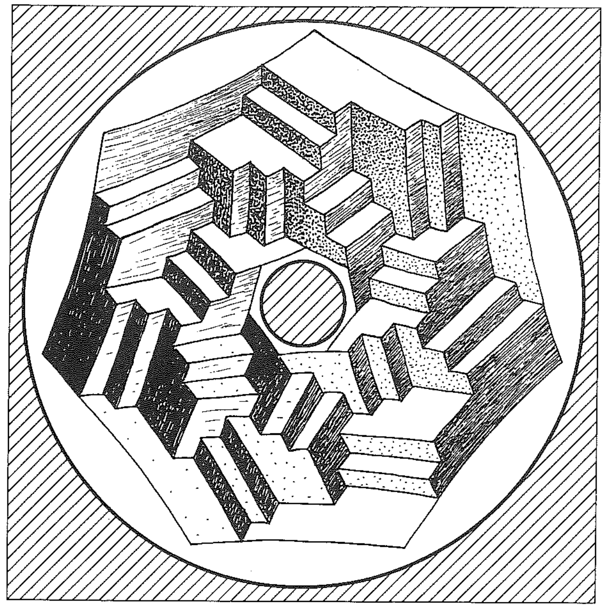


Fig. 3. A more subtle impossible figure with local $Z_2$ ambiguity, drawn on an annular region of the plane.

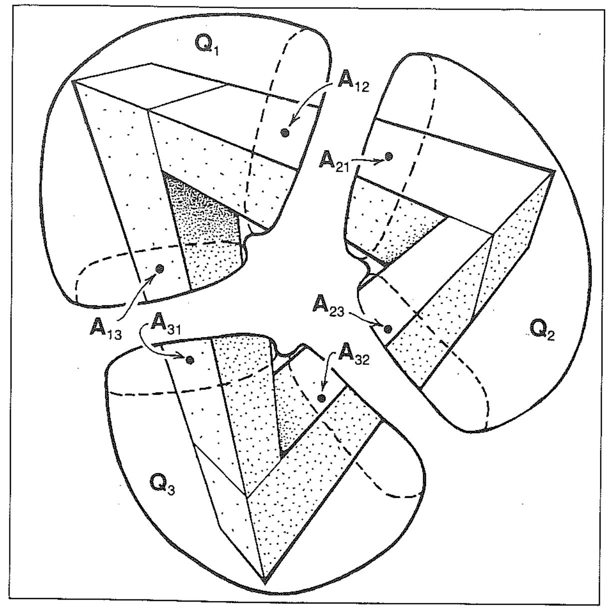



Fig. 4. The tribar shown pieced together out of overlapping smaller drawings, each of which depicts a possible structure.

course there are always other ambiguities, such as the fact that the picture could be depicting a *picture* of another picture, for example, rather than a three-dimensional object (a feature that Escher often put to paradoxical use, e.g. in his lithograph *Drawing Hands* and woodcut *Three Spheres I*). I am excluding this and other possible ambiguities here by my use of the phrase 'natural interpretation'. This distance can be described in terms of positive real numbers d, the set of all possible real numbers being denoted by $\mathbf{R}^+$. I am thinking of $\mathbf{R}^+$ as a multiplicative (Abelian) group, so in this case we have the ambiguity group $G = \mathbf{R}^+$. Let us see how this comes about.

Consider the portion of the figure drawn in region $Q_1$, and fix a point $A_{12}$ on this portion where it overlaps with $Q_2$, and a point $A_{13}$ on it where it overlaps with $Q_3$. Let $A_{21}$ be that point of the figure, as drawn on $Q_2$, that is to be matched with the point $A_{12}$ on $Q_1$, and similarly let $A_{31}$ be the point on $Q_3$ that is to be matched with $A_{13}$. Finally fix a point $A_{23}$ on the part of the figure on $Q_2$ that is to be pasted on $Q_3$, and the corresponding point $A_{32}$ on $Q_3$ that is to be matched with it. See Fig. 4 for the entire arrangement of points.

Let us suppose that there is an actual three-dimensional object $O_1$, which the drawing on $Q_1$ depicts and, similarly, actual objects $O_2$ and $O_3$, which the drawings on $Q_2$ and $Q_3$ depict (see Fig. 5). The point on $O_1$ that is depicted by $A_{12}$ may not be the same distance from the observer's eye E as the corresponding point on $O_2$, depicted by $A_{21}$. Let the ratio of these distances be $d_{12}$, and similarly for other pairs of matched points. Thus we have

$$d_{ij} = \frac{\text{distance from E to point on } O_i \text{ depicted by } A_{ij}}{\text{distance from E to point on } O_j \text{ depicted by } A_{ji}} \qquad (2)$$

We note first that $d_{ij}$ does not actually depend on the particular matched pair of points $A_{ij}$, $A_{ji}$, which are chosen on the overlap between $Q_i$ and $Q_j$. We get the same $d_{ij}$ whichever such matching pair we choose. This $d_{ij}$ represents the factor that we must move out by when we pass from $O_j$ to $O_i$ at the region of overlap.

Note also that

$$d_{ij} = 1/d_{ji} \qquad (3)$$

and that if we change our minds about the object $O_i$ that is being depicted in $Q_i$ (i.e. if we change its chosen distance from the observer's eye) then the pair $(d_{ij}, d_{ik})$ is replaced according to

$$(d_{ij}, d_{ik}) \rightarrow (\lambda d_{ij}, \lambda d_{ik}), \qquad (4)$$

for some positive number $\lambda$.

If, instead of the tribar, we had had some drawing of a figure that could be consistently realized in three-dimensional space, then we could have moved the objects $O_1$, $O_2$ and $O_3$ in and out until they all came together as one consistent structure. This amounts to the fact that by rescalings of the above type we can reduce the three ratios $d_{12}$, $d_{23}$ and $d_{31}$ simultaneously to 1. Another way of saying this is that there exist three (positive) numbers $q_1$, $q_2$, $q_3$ such that

$$d_{ij} = q_i/q_j \qquad (5)$$

for each different i, j. In the terminology of cohomology theory, the collection $\{d_{ij}\}$ is, in the general case, referred to as a *cocycle*. If (5) holds, the cocycle is called a *coboundary*. The replacement (4) provides the *coboundary freedom*, and we regard cocycles as *equivalent* if they can be converted to one another under this freedom. Under this equivalence, we obtain the *cohomology group elements*, i.e. the elements of

$$H^1(Q, \mathbf{R}^+). \qquad (6)$$

The coboundaries provide the *unit* element of (6), and we see from the above discussion that the test for whether or not the figure depicted in Q is 'impossible' is whether or not the resulting element of (6) is indeed the unit element.

I have been discussing impossible figures of the kind that I described earlier [2] as 'pure', i.e. for which the only local ambiguity in the figure is the *distance* from the observer's eye



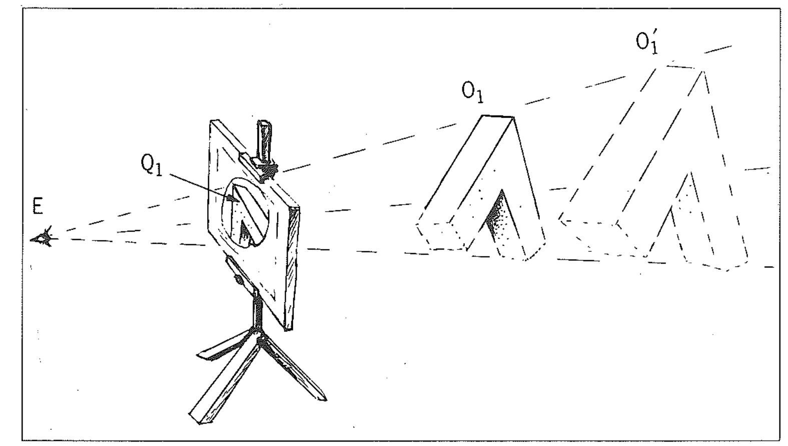



Fig. 5. There is a local $\mathbf{R}^+$ ambiguity in any plane drawing as to the distance from the observer's eye to the object depicted.

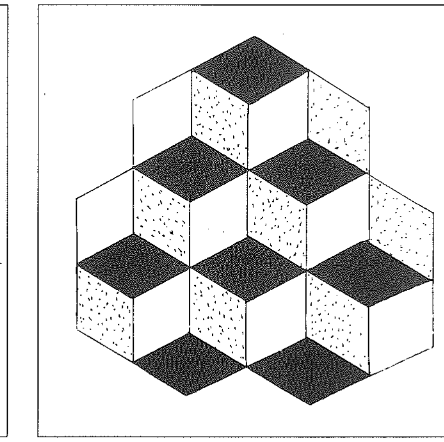


Fig. 6. Necker cubes, with $Z_2$ ambiguity.

of the object being depicted. Often there are other ambiguities of relevance. For the type of impossible figure depicted in Fig. 3, the relevant ambiguity is that of the 'Necker cube', see Fig. 6. Here the ambiguity is just a twofold one, and we can use the numbers +1 and −1 in place of the distance ratios $d_{ij}$ defined in (2), where +1 means that the depicted three-dimensional object $O_i$ *agrees* with $O_j$ where the drawings overlap, and −1 means that the objects *disagree*. The discussion proceeds exactly as before, except that $d_{ij}$, $\lambda$ and $q_i$ now all belong to $\mathbf{Z}_2$ (the multiplicative group consisting of +1 and −1 alone), and the cohomology group element we obtain belongs to

$$H^1(Q, \mathbf{Z}_2). \qquad (7)$$

If we cut Fig. 3 into three pieces analogous to those of Fig. 4 and follow the corresponding procedure through, we indeed find an element of (7) that is *not* the unit element, whereas if Fig. 3 has been drawn 'consistently' (e.g. with a hexagon—or, indeed, an octagon—at the centre, rather than a heptagon), then the unit element would have been obtained. I leave the detailed verification of these facts to the interested reader.

More complicated figures with 'multiple impossibilities' [3] can also be analyzed in this way, but for this we should require a more complete description of what a (Cech) cohomology group actually is. In general, the figure would need to be divided up into more than three pieces, but the essential idea is the same as before [4]. I believe that considerations such as these may open up intriguing possibilities for further exotic types of impossible figure. I hope to be able to consider such matters at a later date.

## References


1. R. Penrose, "Escher and the Visual Representation of Mathematical Ideas", in H. S. M. Coxeter, M. Emmer, R. Penrose and M. L. Teuber, eds., *M. C. Escher: Art and Science* (Amsterdam: North Holland, 1986) pp. 143–147.

2. See Penrose [1].

3. See, for example, L. S. Penrose and R. Penrose, "Impossible Objects: A Special Type of Visual Illusion", *Brit. J. Psych.* **49** (1958) pp. 31–33.

4. For further information, see P. Griffiths and J. Harris, *Principles of Algebraic Geometry* (New York: Wiley, 1978) p. 34.