Math 427/527: algebraic topology
Homework problem for Lecture 1.

Let $\mathsf{Free}\colon \mathsf{Set} \to \mathsf{abGrp}$ be the functor defined by $\mathsf{Free}(X) = \bigoplus_X \mathbb{Z}$ and let $\mathsf{Forget}\colon \mathsf{abGrp} \to \mathsf{Set}$ be the functor defined by forgetting the group structure, that is, $\mathsf{Forget}(G)$ is underlying set of G.

(a) The operations $\times$ and $\otimes$ turn Set and abGrp, respectively, into symmetric monoidal categories. Show that Free and Forget are adjoint monoidal functors. (Note that the main work here is writing down a definition for adjoint monoidal functors, and verifying that this gives an example.)

(b) Now consider *pointed* sets Set_* and $\mathsf{Free}_*(X) = \mathbb{Z}X/\mathbb{Z}x_0$ where $\mathsf{Free}(X) = \mathbb{Z}X$ and $x_0 \in X$ is the basepoint. If $\mathsf{Forget}_*(G)$ is the pointed set with basepoint the identity, show that the appropriate product of pointed sets is given by

$$X \wedge Y = \frac{X \times Y}{(X \times \{y_0\} \cup \{x_0\} \times Y)}$$

and verify the associated adjunction of monoidal functors.