Math 309: Introduction to knot theory

Assignment 4, due Friday November 1 by 10:59 pm.

1. We used $\mathbb{Z}[t, t^{-1}]$ to denote the set of Laurent polynomials in t with integer coefficients. Elements of this set are of the form

$$\sum_{k \in K} c_k t^k$$

where $c_k \in \mathbb{Z}$ and K is a *finite* subset of $\mathbb{Z}$. During lecture, we recalled that polynomials of this form can be added and multiplied following familiar rules.

(a) Check that $(t^{-2} + t^{-4} + t^{-6} + \cdots)$ is a multiplicative inverse for the element $(t^2 - 1)$. Is $t^2 - 1$ invertible in $\mathbb{Z}[t, t^{-1}]$? Justify your answer.

(b) To proceed more systematically, let $A, B \in \mathbb{Z}[t, t^{-1}]$ be arbitrary elements, so there are finite subsets of integers I and J, and for every $i \in I$ and for every $j \in J$, there are integers a_i and b_j such that

$$A = \sum_{i \in I} a_i t^i \qquad B = \sum_{j \in J} b_j t^j$$

The product AB is an element $C \in \mathbb{Z}[t, t^{-1}]$, so there is a finite subset $K \subset \mathbb{Z}$ and integers c_k such that

$$C = \sum_{k \in K} c_k t^k$$

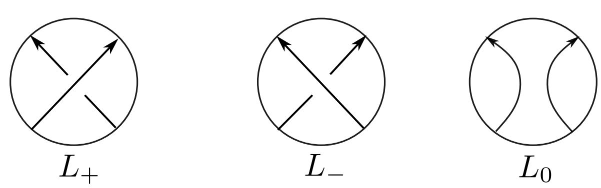
Express the coefficients c_k in terms of the coefficients a_i and b_j.

(c) Use part **(b)** to prove that if $AB = 1$ then $A = \pm t^n$, for some $n \in \mathbb{Z}$.

2. According to Adams (see Exercise 6.8), the Jones polynomial satisfies the "skein relation"

$$t^{-1} V(L_+) - t V(L_-) = (t^{1/2} - t^{-1/2}) V(L_0), \tag{1}$$

where L_+, L_-, L_0 are oriented diagrams of links that are the same everywhere except for the interior of a circle, inside of which they differ as follows:



Prove that any polynomial invariant $p_L(t) \in \mathbb{Z}[t, t^{-1}]$ of oriented links that satisfies the skein relation (1), and takes the value $p_{\text{unknot}}(t) = t^{-1} + t$, has the property that $p_{L^*}(t) = p_L(t^{-1})$ where L^* is the mirror of L.

3. Let T_n denote the rational tangle associated with the integer n, so that T_0 and T_∞ represent the two crossingless tangles.

(a) Using an inductive argument, find an expression for $\langle T_n \rangle$ in terms of $\langle T_0 \rangle$ and $\langle T_\infty \rangle$ that is valid for all choices of integer n.

(b) Using your result from part **(a)**, calculate $\langle K \rangle$ and $V_K(t)$ when K is the figure-eight knot. Show all your work.

(c) Using your result from part **(a)**, calculate $\langle K \rangle$ when K is the knot 8_5 (the appendix in Adams gives a diagram for this knot). Show all your work.