**ON SEPARATING MUTANT PAIRS**

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We recently posted a paper explaining how, using a refinement of Khovanov cohomology, it is possible to separate mutant pairs of knots in the presence of an appropriate involution. We elaborate on these examples here.

The refinement in question associates with a knot $K$ and a fixed strong inversion a triply graded finite dimensional vector space $\widetilde{\mathrm{Kh}}_\tau(K)$ [3]. These integer gradings are denoted by $i, j, k$, where $i, j$ arise from filtrations associated to the usual bigradings on Khovanov cohomology. The $k$-grading is, on the other hand, tightly connected with the strong inversion. We proved:

**Theorem 1.2.** *There exist pairs of knots related by mutation, with identical Khovanov cohomology and identical knot Floer homology, which can be distinguished from one another by the triply graded refinement of Khovanov cohomology by appealing to an appropriate symmetry.*

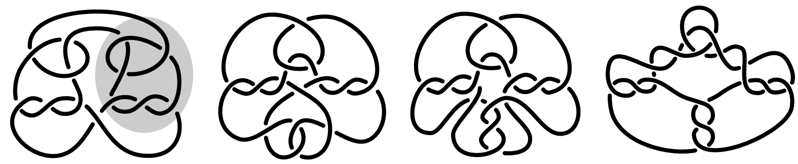

**Figure 1.** Strongly invertible diagrams $D$, $D_1^\mu$, $D_2^\mu$, and $D_3^\mu$ (from left to right). The diagram $D$, representing the knot $K$, highlights a tangle in the projection; rotating this tangle 180 degrees in the plane yields the mutant knot of interest, denoted $K^\mu$. Three diagrams for $K^\mu$ are shown; $D_1^\mu$ and $D_2^\mu$ are equivalent as involutive diagrams [3, Definition 2.2] however, as we will show, $D_3^\mu$ is a distinct strongly invertible diagram.

The proof is constructive. With this brief note, which will eventually be incorporated into [3], we aim to clarify (and simplify) the construction used and to make explicit how infinite families of such pairs can be readily obtained.

For concreteness, consider the knots $K$ and $K^\mu$ described in Figure 1. Since $K$ is an alternating (non-torus) knot, it follows that $K$ is hyperbolic; and since mutation preserves the property of being alternating, $K^\mu$ is alternating as well. This observation allows us to conclude that $K$ and $K^\mu$ have identical knot Floer homology, since this invariant is determined by the Alexander polynomial and the knot signature for alternating knots. Since $K$ and $K^\mu$ are related by mutation, it is also immediate that the Khovanov cohomologies agree $\widetilde{\mathrm{Kh}}(K) \cong \widetilde{\mathrm{Kh}}(K^\mu)$ as bigraded vector spaces [1, 4].

Now observe that on any given hyperbolic knot, there are at most two strong inversions [2, Proposition 3.1]. Therefore if we can show that $D$, $D_1^\mu$ (equivalently, $D_2^\mu$), and $D_3^\mu$ have mutually distinct $k$-grading support, it follows that $K$ and $K^\mu$ are not isotopic. It is a somewhat striking

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| $K$ | | $K^\mu$ | | | | | |
|---|---|---|---|---|---|---|---|
| $D$ | | $D_1^\mu$ | | $D_2^\mu$ | | $D_3^\mu$ | |
| $i_{\min}=-4$ | $k_{\min}=-2$ | $i_{\min}=-4$ | $k_{\min}=-3$ | $i_{\min}=-5$ | $k_{\min}=-4$ | $i_{\min}=-7$ | $k_{\min}=-4$ |
| $i_{\max}=9$ | $k_{\max}=5$ | $i_{\max}=10$ | $k_{\max}=6$ | $i_{\max}=9$ | $k_{\max}=5$ | $i_{\max}=9$ | $k_{\max}=6$ |

**Table 1.** A summary of calculations: choosing orientations on each of the diagrams of Figure 1 determines the maximal and minimal $i$- and $k$-gradings of non-zero homogeneous summands of the cochain complexes in each case. Bigradings in which the cohomology $\widetilde{\mathrm{Kh}}_\tau$ is non-trivial, according to the support lemma, are shaded.

fact that this can be shown by appealing only to the diagrams shown, namely, without complete calculation of the triply-graded cohomology theory in each case.

Given an oriented involutive diagram $D$ we we denote the number of positive crossings in $D$ by $n_+(D) = n_+^A(D) + n_+^E(D)$ where $n_+^A$ counts on-axis positive crossings and $n_+^E$ counts off-axis positive crossings. Similarly, $n_-(D) = n_-^A(D)+n_-^E(D)$ where $n_-^A$ counts on-axis negative crossings while $n_-^E$ counts off-axis negative crossings. We write

$$k_{\min}(D) = -\tfrac{1}{2}n_-^E(D) - n_-^A(D) \quad \text{and} \quad k_{\max}(D) = \tfrac{1}{2}n_+^E(D) + n_+^A(D)$$

for the minimum and maximum $k$-gradings, respectively, in which the involutive cochain complex $\widetilde{\mathrm{CKh}}_\tau(D)$ is non-zero. These provide lower and upper bounds, respectively, on the possible $k$-gradings of non-zero homogeneous summands of $\widetilde{\mathrm{Kh}}_\tau(K)$. This is a direct analogue of the fact that

$$i_{\min}(D) = n_-(D) \quad \text{and} \quad i_{\max}(D) = n_+(D)$$

provide *a priori* bounds on the $i$-grading of $\widetilde{\mathrm{Kh}}(K)$. Furthermore we have the following:

**Lemma 6.1 (The support lemma).** *Fix an involutive diagram $D$ with $m$ negative crossings and $n$ positive crossings for some strongly inversion of a knot $K$. If $\widetilde{\mathrm{Kh}}(K)$ is non-trivial in grading $i=-m$ then $\widetilde{\mathrm{Kh}}_\tau(K)$ is non-trivial in bigrading $(i,k)=(-m,k_{\min}(D))$ and, similarly, if $\widetilde{\mathrm{Kh}}(K)$ is non-trivial in grading $i=n$ then $\widetilde{\mathrm{Kh}}_\tau(K)$ in non-trivial in bigrading $(i,k)=(n,k_{\max}(D))$.*

For the specific diagrams in question, the calculation of $k_{\max}$ and $k_{\min}$ is summarized in Table 1. Note that, since $K$ is alternating, we know that $\widetilde{\mathrm{Kh}}(K)$ is non-trivial in $i$-grading 9 and in $i$-grading $-4$. As a result of these calculations, we conclude that the $k$-grading is sufficient to distinguish 3 strong inversions between the knots $K$ and $K^\mu$; it follows that $K$ and $K^\mu$ must be distinct knots, since neither admits more than 2 strong inversions.

We remark that, based on the diagrams shown, it is possible to determine dihedral subgroups $D_{2\ell}$ and $D_{2\ell_\mu}$ in the symmetry groups of $K$ and $K^\mu$, respectively (where the positive integers $\ell$ and $\ell_\mu$ may be distinct): each knot admits a periodic symmetry of order 2. However, based on the tools we have presented here, neither $\ell$ nor $\ell_\mu$ is determined.

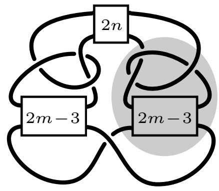


**Figure 2.** Infinite families.

There are various ways in which the particular choice $D$ can be altered to produce infinite families of mutant pairs amenable to being distinguished by our invariant; we give two. Let $n$ be any non-negative integer and let $m$ be any non-positive integer. Consider the diagram shown in Figure 2; $m=n=0$ recovers $D$. Our convention is that $+1$ is the crossing $\times$ so that the resulting knot is alternating for all choices $m, n$. The reader can check that the mutation and the isotopies giving rise to the diagrams in Figure 1 carry through for arbitrary $m$ and $n$.

Finally, it is worth noting that the restriction to alternating knots is not an essential one. The restriction rather allowed us to extract information from the triply-graded invariant without having to calculate the invariant in full; easily determine properties of Khovanov cohomology and knot Floer homology; and to conclude that the examples were hyperbolic without appeal to additional machinery. However, similar constructions are possible in which the knots are not alternating, and it seems likely that calculating the full invariant in these settings would yield similar results.

## References

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