ANY TANGLE EXTENDS TO NON-MUTANT KNOTS WITH THE SAME JONES POLYNOMIAL

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ABSTRACT. We show that an arbitrary tangle T can be extended to produce diagrams of two distinct knots that cannot be distinguished by the Jones polynomial. When T is a prime tangle, the resulting knots are prime. It is also shown that, in either case, the resulting pair are not mutants.

1. INTRODUCTION

The following question is still unanswered: Is there a nontrivial knot with Jones polynomial 1? This question has motivated a range of tools for generating pairs of distinct knots sharing a common Jones polynomial [1, 7, 14, 15]. The prototype for these is the construction of mutation due to Conway [4], and it is well known that this technique preserves the HOMFLY polynomial [7]. However, it can be shown that the mutant of any diagram of the unknot is always unknotted [14, 15]. We take this as motivation for the following:

Theorem. *For any prime tangle T, there exits a pair of distinct prime knots (each containing T in their diagram) that cannot be distinguished by the Jones polynomial. Moreover, these knots are not related by mutation.*

For links having 2 or 3 components, Thistlethwaite [17] has found examples of non-trivial links having trivial Jones polynomial. It is shown in [5] that there are n-component non-trivial links having trivial Jones polynomial for all $n > 1$. We present some new examples of non-trivial knots that cannot be distinguished by the Jones polynomial. The methods used here combine ideas from Eliahou, Kauffman and Thistlethwaite [5] and Kanenobu [10].

2. POLYNOMIALS

To establish notation, we review the construction of the Jones polynomial and its 2-variable generalization, the HOMFLY polynomial. Let $\Lambda = \mathbb{Z}[a, a^{-1}]$ and K be a knot, that is a smooth or piecewise linear embedding $\mathbb{S}^1 \hookrightarrow \mathbb{S}^3$. We will confuse the knot and a diagram representing it, denoting both by K. More generally, an n-component link L is a collection of n disjointly embedded circles $\amalg_{i=1}^n \mathbb{S}^1_i \hookrightarrow \mathbb{S}^3$ [16].

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We recall that the Kauffman bracket $\langle L \rangle \in \Lambda$ of a link L is obtained recursively from the axioms [11]

(1) $$\langle \bigcirc \rangle = 1$$

(2) $$\langle \times \rangle = a \langle = \rangle + a^{-1} \langle \supset \subset \rangle$$

(3) $$\langle L \amalg \bigcirc \rangle = \delta \langle L \rangle$$

where $\delta = -a^{-2} - a^2$. The vignettes $\langle \ \rangle$ indicate that the changes are made to the diagram locally, while the rest of diagram is left unchanged. The Jones [8, 9] polynomial $V_L(a) \in \Lambda$ may be defined by

$$V_L(a) = (-a^{-3})^{w(\vec{L})} \langle L \rangle$$

where the writhe $w(\vec{L}) \in \mathbb{Z}$ is obtained by assigning an orientation to L, and taking a sum over all crossings of L by a right hand rule:

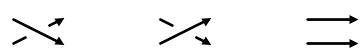
$$w(L_+) = 1 \qquad w(L_-) = -1.$$

For the HOMFLY [6] polynomial $P_L(t, x)$, a similar recursive definition exists:

(4) $$P_{\bigcirc}(t, x) = 1$$

(5) $$t^{-1} P_{L_+}(t, x) - t P_{L_-}(t, x) = x P_{L_0}(t, x)$$

In this setting, L_+, L_- and L_0 are diagrams that are identical except in a small region where they differ as in



L_+ $\quad$ L_- $\quad$ L_0 .

With this notation, the Jones polynomial may be recovered from the HOMFLY polynomial by specifying

$$V_L(a) = P_L\left(a^{-4}, a^{-2} - a^2\right)$$

where $t = a^{-4}$ recovers the standard form of the Jones polynomial.

3. Tangles

For the purpose of this paper, a tangle T will be given by the intersection $B^3 \cap L$ where $B^3 \subset \mathbb{S}^3$ is a 3-ball, and ∂B^3 intersects the link L transversely in exactly 4 points [4]. Following notation of Lickorish [12], we may denote T by the pair (B^3_T, t), where t are the arcs given by $B^3_T \cap L$. Note that by taking $\mathbb{S}^3 = B^3_T \cup B^3_U$ we can denote $L = T \cup U$ provided $\partial t = \partial u$.

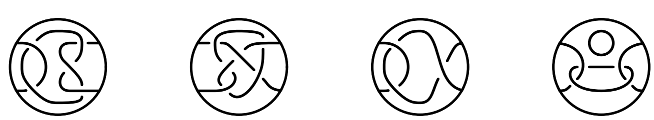
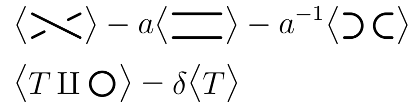
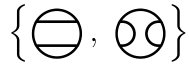


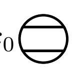
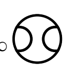
FIGURE 1. Some diagrams of tangles.

Again, we will confuse a tangle T and a diagram representing it, noting that a diagram of a tangle is obtained by intersecting a disk with a diagram for L. Equivalence of tangles, as subsets of link diagrams, is given by isotopy fixing the four boundary points, together with the Reidemeister moves [13, 16].

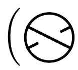
Let $\mathcal{M}$ be the free Λ-module generated by equivalence classes of tangles, and let $\mathcal{I}$ be the ideal generated by the elements



The Kauffman bracket skein module $\mathcal{S} = \mathcal{M}/\mathcal{I}$ is generated by the tangles  denoted 0 and ∞ respectively [14, 15]. For $T \in \mathcal{S}$ we have

$T = x_0$  $+ x_\infty$  $= \begin{bmatrix} x_0 & x_\infty \end{bmatrix} \begin{bmatrix} 0 \\ \infty \end{bmatrix}$

where $x_0, x_\infty \in \Lambda$. $br(T) = \begin{bmatrix} x_0 & x_\infty \end{bmatrix}$ is called the bracket vector of T [5]. For example,

$br\Big($  $\Big) = \begin{bmatrix} a & a^{-1} \end{bmatrix}.$

Now consider disjoint 3-balls $B^3_T, B^3_U \subset \mathbb{S}^3$ defining tangles T, U in some knot K. Write $K = K(T, U)$ so that

$$\langle K(T,U) \rangle = br(T)\, \mathcal{K}\, br(U)^{\mathrm{t}}$$

defines a bilinear map $\langle K(-,-) \rangle : \mathcal{S} \times \mathcal{S} \to \Lambda$ where

$$\mathcal{K} = \begin{bmatrix} \langle K(0,0) \rangle & \langle K(0,\infty) \rangle \\ \langle K(\infty,0) \rangle & \langle K(\infty,\infty) \rangle \end{bmatrix}.$$

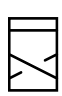
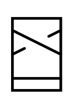
This will be referred to as the evaluation matrix for $K(T, U)$.

Two tangles T, U are homeomorphic [12] if there is a homeomorphism of pairs $(B^3_T, t) \to (B^3_U, u)$. Note that this homeomorphism need not be the identity on the boundary; the tangles 0 and ∞ are homeomorphic, for example. In general, a tangle is rational if it is homeomorphic to 0.

A tangle T is prime [12] whenever it is non-rational, t is a pair of disjoint arcs, and any $\mathbb{S}^2 \subset B^3_T$ meeting t transversely in 2 points bounds a ball in B^3_T containing an unknotted arc. In figure 1 for example, the first and second tangles from the left are prime (see [12]).

4. Braid Actions

Let $B_3 = \langle \sigma_1, \sigma_2 | \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$ be the three strand braid group [2, 3] with standard generators

$\sigma_1 =$  $\qquad \sigma_2 =$ .

Given $T \in \mathcal{S}$ and $\beta \in B_3$ we can define a new tangle denoted T^β as in figure 2, and it is easy to check that this is a well defined group action $\mathcal{S} \times B_3 \to \mathcal{S}$. Note that T and T^β are homeomorphic tangles. In particular, if T is prime, then so is T^β. Let

$$\Sigma_1 = \begin{bmatrix} -a^{-3} & 0 \\ a^{-1} & a \end{bmatrix} \qquad \Sigma_2 = \begin{bmatrix} a & a^{-1} \\ 0 & -a^{-3} \end{bmatrix}$$

and define a group homomorphism $\Phi : B_3 \longrightarrow GL_2(\Lambda)$ via $\Phi(\sigma_j) = \Sigma_j$. A direct computation gives the following (see also [5]):

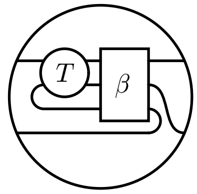


FIGURE 2. The tangle T^β.

Proposition 4.1. *For $T \in \mathcal{S}$, $br(T^{\sigma_j}) = br(T)\Sigma_j$.*

Remark 4.2. *It is possible to extend this action to 4-braids, and Φ to a homomorphism $\Phi : B_4 \to GL_2(\Lambda)$. However, $\Phi(\sigma_1) = \Phi(\sigma_3)$ since the tangles T^{σ_1} and T^{σ_3} are related by mutation and a flype.*

Now for $\beta \in B_3$, consider the linear transformation defined by

$$\begin{aligned} \beta : \mathcal{S} \times \mathcal{S} &\longrightarrow \mathcal{S} \times \mathcal{S} \\ (T, U) &\longmapsto (T^\beta, U^{\beta^{-1}}). \end{aligned}$$

For a knot $K = K(T, U)$, this leads to the definition of a new knot $K^\beta = K(T^\beta, U^{\beta^{-1}})$. Note that

$$\langle K^\beta \rangle = br(T)\Phi(\beta)\,\mathcal{K}\,(\Phi(\beta^{-1}))^{\mathrm{t}} br(U)^{\mathrm{t}}.$$

Whenever $\mathcal{K} \in GL_2(\Lambda)$ (as is the case in the examples considered in the following sections), we define a second B_3-action

$$\begin{aligned} B_3 \times GL_2(\Lambda) &\longrightarrow GL_2(\Lambda) \\ (\beta, \mathcal{K}) &\longmapsto \Phi(\beta)\,\mathcal{K}\,(\Phi(\beta^{-1}))^{\mathrm{t}}. \end{aligned}$$

Under this action, when $\beta \in B_3$ gives rise to a fixed point, the linear transformation given by β fits into the commutative diagram

$$\begin{array}{ccc} \mathcal{S} \times \mathcal{S} & \xrightarrow{\langle K(-,-) \rangle} & \\ {\scriptstyle\beta}\big\downarrow & & \Lambda\,. \\ \mathcal{S} \times \mathcal{S} & \xrightarrow[\langle K(-,-) \rangle]{} & \end{array}$$

In particular, whenever $\mathcal{K} \in \mathrm{Fix}(\beta)$ it follows that $\langle K \rangle = \langle K^\beta \rangle$ and we would like to study the case where K and K^β are distinct knots.

5. KANENOBU KNOTS

Shortly after the discovery of the HOMFLY polynomial, Kanenobu introduced families of distinct knots having the same HOMFLY polynomial and hence the same Jones polynomial [10]. We extend these examples to define a larger class of Kanenobu knots as in figure 3. These will be denoted by $K(T, U)$ for tangles T, U. Another direct computation gives the following:

Proposition 5.1. *Suppose $x \in \Lambda$ so that*

$$\mathcal{X} = \begin{bmatrix} x & \delta \\ \delta & \delta^2 \end{bmatrix} \in GL_2(\Lambda).$$

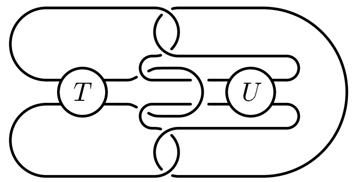


FIGURE 3. The Kanenobu knot $K(T,U)$ for tangles T,U.

Then $\Phi(\sigma_2)\,\mathcal{X}\,\Phi(\sigma_2^{-1})^{\mathrm{t}} = \mathcal{X}$ *and* $\mathcal{X} \in \mathrm{Fix}(\sigma_2)$ *under the* B_3*-action on* $GL_2(\Lambda)$*.*

Since $K(0,0)$ is the knot $4_1\#4_1$, we can compute the evaluation matrix

$$\mathcal{K} = \begin{bmatrix} (a^{-8} - a^{-4} + 1 - a^4 + a^8)^2 & \delta \\ \delta & \delta^2 \end{bmatrix}$$

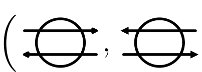
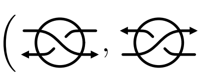
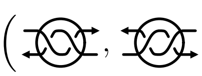
for the Kanenobu knot $K(T,U)$. This $\mathcal{K}$ is of the form given in proposition 5.1, so it follows that

Lemma 5.2. *Whenever tangles* T,U *are chosen such that* $w(K) = w(K^{\sigma_2})$*, the family of knots given by* $K(T^{\sigma_2^n}, U^{\sigma_2^{-n}})$ *are indistinguishable by the Jones polynomial for* $n \in \mathbb{Z}$*.*

Remark 5.3. *While we are only making use of an action of* B_2 *in this setting, an example of an application of* B_3 *can be found in* [5] *where the operator* ω *is the braid* $\sigma_2^2\sigma_1^{-1}\sigma_2^2$*.*

6. BASIC EXAMPLES

Consider the Kanenobu knots

$K_0 = K\Big($  $\Big)$ $\quad K_1 = K\Big($  $\Big)$ $\quad K_2 = K\Big($  $\Big)$

and notice that by applying the action of σ_2 we have

$$K_0 \xrightarrow{\sigma_2} K_1 \xrightarrow{\sigma_2} K_2$$

so that by construction, these knots have the same Jones polynomial

$$a^{-16} - 2a^{-12} + 3a^{-8} - 4a^{-4} + 5 - 4a^4 + 3a^8 - 2a^{12} + a^{16}.$$

On the other hand

$$\begin{aligned} P_{K_0}(t,x) &= (t^{-4} - 2t^{-2} + 3 - 2t^2 + t^4) + (-2t^{-2} + 2 - 2t^2)x^2 + x^4 \\ P_{K_1}(t,x) &= (2t^{-2} - 3 + 2t^2) + (3t^{-2} - 8 + 3t^2)x^2 + (t^{-2} - 5 + t^2)x^4 - x^6 \\ P_{K_2}(t,x) &= (t^{-4} - 2t^{-2} + 3 - 2t^2 + t^4) + (-2t^{-2} + 2 - 2t^2)x^2 + x^4 \end{aligned}$$

and we can conclude that K_0 and K_1 (or K_1 and K_2) are distinct knots. Moreover, these knots cannot be mutants as they have different HOMFLY polynomials. Notice that the equality $P_{K_0} = P_{K_2}$ is consistent with Kanenobu's results [10].

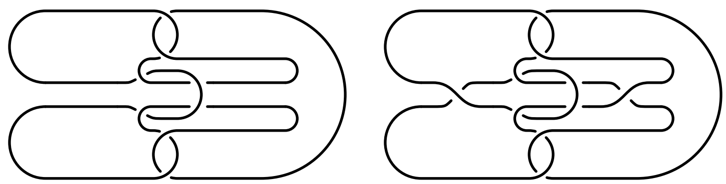


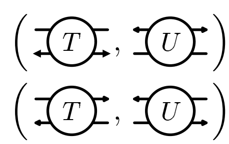
FIGURE 4. Distinct, non-mutant knots with identical Jones polynomial.

7. PROOF OF THE THEOREM

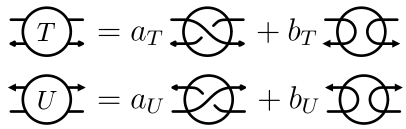
For any tangle T, choose U such that $w(U) = -w(T)$ by switching each crossing of T and define Kanenobu knots $K = K(T, U)$ and $K^{\sigma_2} = K(T^{\sigma_2}, U^{\sigma_2^{-1}})$ (see figure 6). From lemma 5.2 we have that $V_K = V_{K^{\sigma_2}}$. To see that these are distinct knots, we compute the HOMFLY polynomials P_K and $P_{K^{\sigma_2}}$. The requirement on the tangle U gives two choices of orientations for the tangles that are compatible with an orientation of the knot (or possibly link, in which case a choice of orientation is made) $K(T, U)$. They are

Type 1

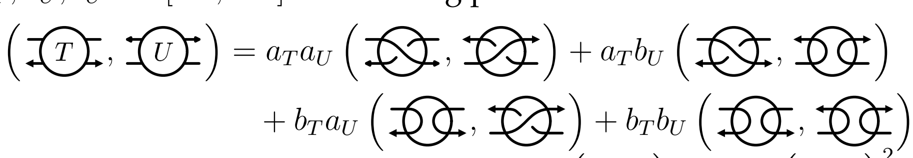
Type 2



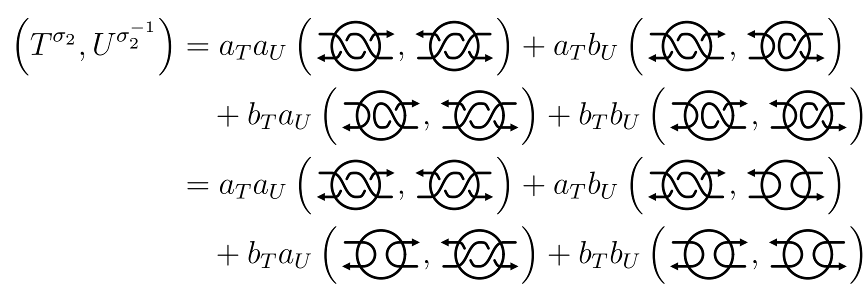
so we proceed in two cases. For knots of type 1 we use the skein relation (5) to decompose



where $a_T, b_T, a_U, b_U \in \mathbb{Z}[t^{\pm 1}, x^{\pm 1}]$. Combining pairwise we obtain



so that $P_K = a_T a_U P_{K_1} + R$ where $R = (a_T b_U + b_T a_U)\left(\frac{t^{-1}-t}{x}\right) + b_T b_U \left(\frac{t^{-1}-t}{x}\right)^2$. Now applying the action of σ_2 we have



so that $P_{K^{\sigma_2}} = a_T a_U P_{K_2} + R$. Since $P_{K_1} \neq P_{K_2}$ (see previous section), we have that $P_K \neq P_{K^{\sigma_2}}$ giving rise to distinct knots. A similar procedure applies for type 2 tangles and is left to the reader. As the knots constructed have different HOMFLY polynomials, we conclude that they cannot be mutants despite having identical Jones polynomial.

$K(T, U)$ can be viewed as a $T \cup V$, where V is the tangle given in figure 5. It follows from [12] that $K(T, U)$ is prime whenever both T and V are prime tangles. Whenever T is prime, U is prime also, therefore it remains to show that V is prime.

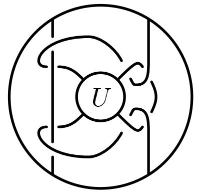


FIGURE 5. The tangle $V = (\mathbb{S}^3 \setminus B^3_T, \text{v})$.

Choosing $U = 0$ we see that each arc of V is unknotted, hence there is no knotted arc-ball pair. In this case closing V to obtain $4_1\#4_1$ yields bridge index 3. Since the union of two rational tangles is a 2-bridge knot [12], we conclude that V must be a prime tangle. Applying lemma 2 of [12], V is a prime tangle for any choice of prime tangle U.

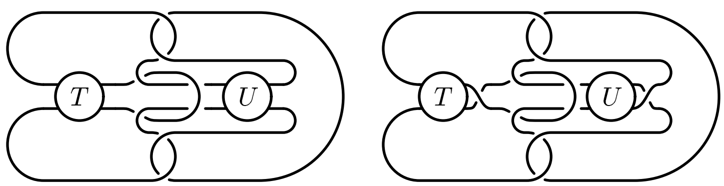


FIGURE 6. Distinct, non-mutant knots with identical Jones polynomial for arbitrary tangle T.

8. FURTHER EXAMPLES

Taking β to be the braid $\sigma_1^2\sigma_2^{-1}\sigma_1\sigma_5^{-2}\sigma_4\sigma_5^{-1} \in B_6$, the Kanenobu knots have the alternate diagram given in figure 7. For general $\beta \in B_6$, we can define a closure $L_\beta(T, U)$ in this

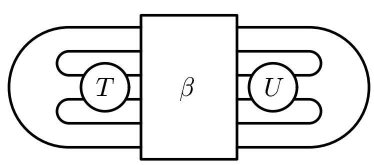


FIGURE 7. The link $L_\beta(T, U)$.

way to obtain a link for tangles T, U. Define the subgroup $G < B_6$ via the composite homomorphism

$$\begin{array}{ccccc} B_3 & \longrightarrow & B_3 \oplus B_3 & \longrightarrow & B_6 \\ \alpha & \longrightarrow & (\alpha, \alpha) & \longrightarrow & i_0(\alpha)i_3(s\alpha) \end{array}$$

where $i_k : B_3 \to B_6$ via $i_k(\sigma_j) = \sigma_{j+k}$, and $s : B_3 \longrightarrow B_3$ by $s\sigma_1 = \sigma_2^{-1}$ and $s\sigma_2 = \sigma_1^{-1}$. From this construction it follows that the evaluation matrix for any link of the form $L_\beta(T, U)$ is of the form given in proposition 5.1 whenever $\beta \in G$. We may revisit the argument in the previous section with a link of the form $L_\beta(T, U)$, and obtain a range of new examples of links that cannot be distinguished by the Jones polynomial.

For example, let $\beta = \sigma_1^3\sigma_2^{-1}\sigma_1\sigma_5^{-3}\sigma_4\sigma_5^{-1} \in G$ then $K = L_\beta(0,0)$ is the knot $5_2\#5_2^\star$. Our construction shows that K^{σ_2} and K have the same Jones polynomial

$$-a^{-20} + 2a^{-16} - 4a^{-12} + 6a^{-8} - 7a^{-4} + 9 - 7a^4 + 6a^8 - 4a^{12} + 2a^{16} - a^{20},$$

however these knots (illustrated in figure 8) are once again distinguished by the HOMFLY polynomial:

$$\begin{aligned} P_K &= (-4t^{-2} + 9 - 4t^2) + (-8t^{-2} + 20 - 8t^2)x^2 \\ &\quad + (-5t^{-2} + 18 - 5t^2)x^4 + (-t^{-2} + 7 - t^2)x^6 + x^8 \\ P_{K^{\sigma_2}} &= (-t^{-4} + 3 - t^4) + (-t^{-4} + t^{-2} + 4 - t^2 + t^4 - t^6)x^2 + (t^{-2} + 2 + t^2)x^4 \end{aligned}$$

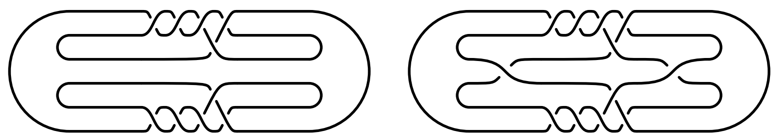


FIGURE 8. The knots $K = L_\beta(0,0)$ and $K^{\sigma_2} = L_\beta(0^{\sigma_2}, 0^{\sigma_2^{-1}})$.

Acknowledgement. This work made use of the program KNOTSCAPE for the HOMFLY polynomial computations. In addition, I wish to thank Dale Rolfsen for helpful discussions, as this work made up part of my M.Sc. thesis under his supervision.

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