Heegaard Floer homology solid tori

Liam Watson

www.math.ucla.edu/~lwatson

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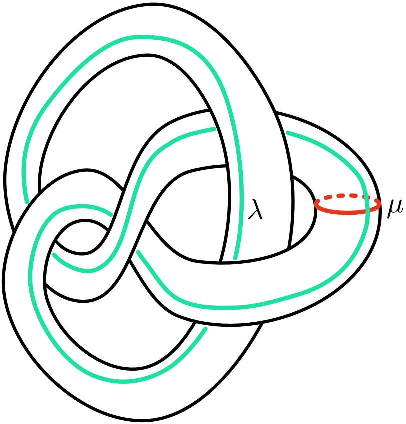
Dehn surgery on knots



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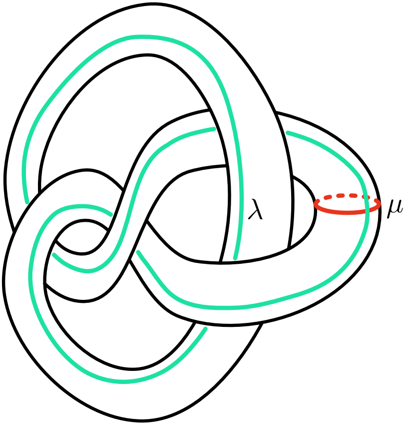

Set $M = S^3 \setminus \nu(K)$ with

- $\langle \mu, \lambda \rangle \cong H_1(\partial M; \mathbb{Z})$
- $\mu \cdot \lambda = +1$

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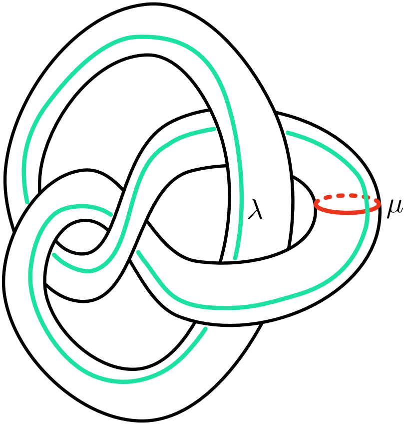

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$$\boxed{S^3_{p/q}(K) = M \cup_h (D^2 \times S^1)}$$

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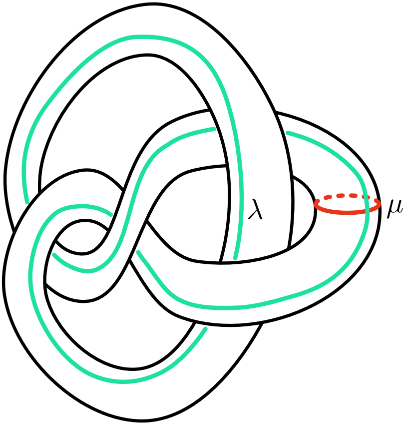

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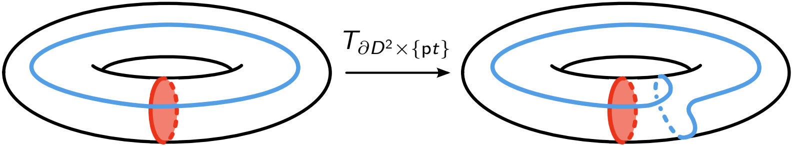
$$S^3_{p/q}(K) = M \cup_h (D^2 \times S^1)$$

$$h : \partial D^2 \times S^1 \to \partial M$$
$$\partial D^2 \times \{\mathrm{pt}\} \mapsto p\mu + q\lambda$$

Alexander's trick

The homeomorphism h extends uniquely to the rest of the solid torus (i.e. the 3-ball).

The Alexander trick



A Dehn twist along $\partial D^2 \times \{\mathrm{pt}\}$ in the boundary of $D^2 \times S^1$ extends to a homeomorphism of the solid torus.

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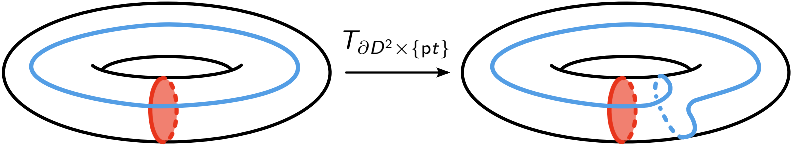


A Dehn twist along $\partial D^2 \times \{\mathrm{pt}\}$ in the boundary of $D^2 \times S^1$ extends to a homeomorphism of the solid torus.

This observation characterizes the solid torus, among orientable, irreducible 3-manifolds with torus boundary, in the following sense:

Theorem (Johannson, see Siebenmann or McCullough)

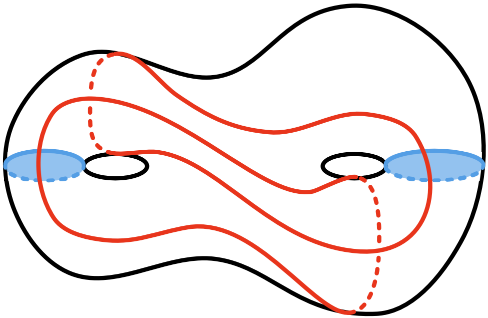
Let M be an orientable, irreducible 3-manifold with torus boundary. If M admits a homeomorphism h for which $h|_{\partial M}$ is a Dehn twist, then $M \cong D^2 \times S^1$.

Decomposing along tori

More generally, one would like to study the closed manifold $M_1 \cup_h M_2$ for a given pair of (orientable) 3-manifolds M_1, M_2 and homeomorphism $h : \partial M_1 \to \partial M_2$.

In this talk, we will consider the decomposition of a closed, orientable 3-manifold Y along an interesting (incompressible, two-sided) torus and study the pieces M_1 and M_2 of $Y = M_1 \cup_h M_2$.

Decomposing along tori (viewed from a Heegaard diagram)

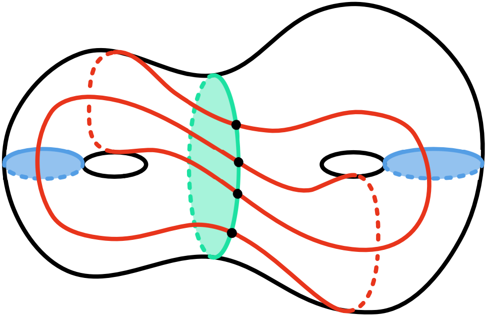


Consider a self indexing Morse function

$$f : Y \to [0,3]$$

The surface $f^{-1}(\frac{3}{2})$ gives rise to a Heegaard diagram $\mathcal{H}$ for Y. Pictured is the inverse image of $[\frac{3}{2}, 3]$; there is a critical point of index 2 in the interior of each blue disk.

Decomposing along tori (viewed from a Heegaard diagram)



Now consider a properly embedded disk in the handlebody $f^{-1}([\frac{3}{2}, 3])$, meeting the red attaching curves transversely in 4 points.

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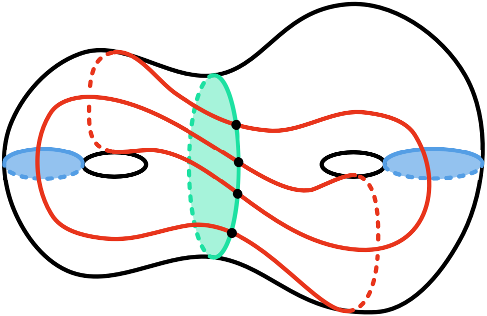

Now consider a properly embedded disk in the handlebody $f^{-1}([\frac{3}{2}, 3])$, meeting the red attaching curves transversely in 4 points.

Claim: The boundary of this disk in $\mathcal{H}$ represents a torus in Y.

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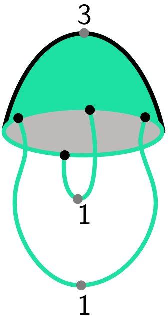
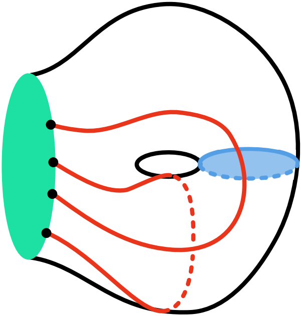
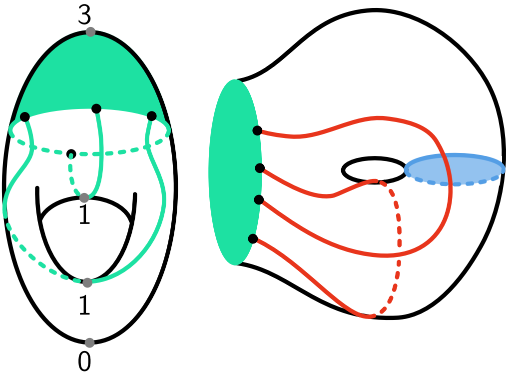




First, suppose that the critical point of index 3 is in the interior of the green disk.
Next, notice that the points of intersection are paired, according to index 1 critical points.

(indices of critical points labeled)

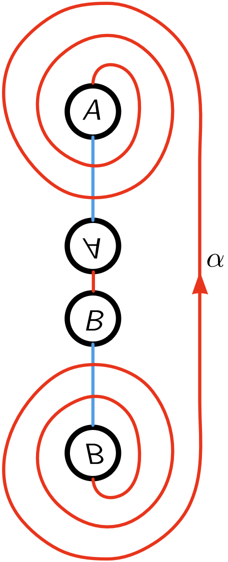
Decomposing along tori (viewed from a Heegaard diagram)



(indices of critical points labeled)

Finally, the remaining points in the boundary of the disk flow to the index 0 critical point.

Another view: bordered Heegaard diagrams



Add a collection of handles A and B to a sphere to obtain a handlebody.

As before, β-curves in blue and α-curves in red.

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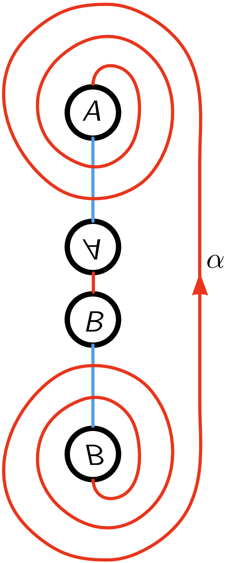

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Exercise

For this particular example:
(1) the manifold has torus boundary
(2) the fundamental group is $\langle a, b | a^2 b^2 \rangle$
(3) the manifold is the twisted I*-bundle over the Klein bottle*

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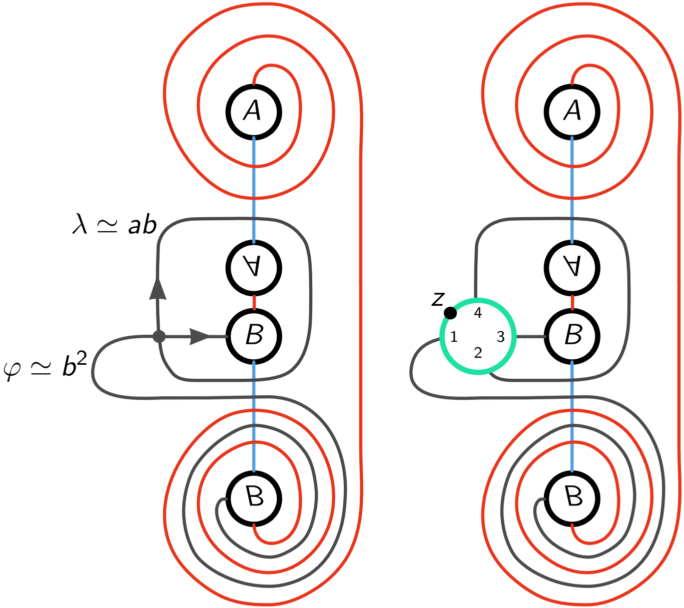

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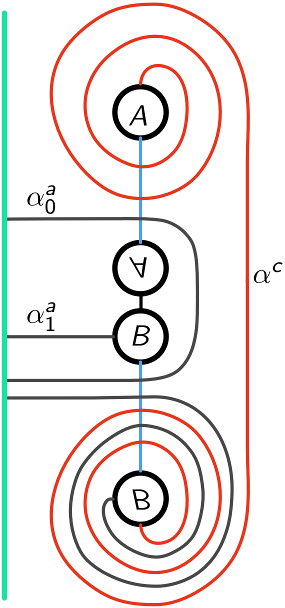

For the purpose of this talk, a bordered manifold is an ordered triple $(M, \alpha_0^a, \alpha_1^a)$ where

- M is a manifold with $\partial M = S^1 \times S^1$,
- $\langle \alpha_0^a, \alpha_1^a \rangle$ generates the peripheral subgroup $\pi_1(\partial M) \subset \pi_1(M)$.

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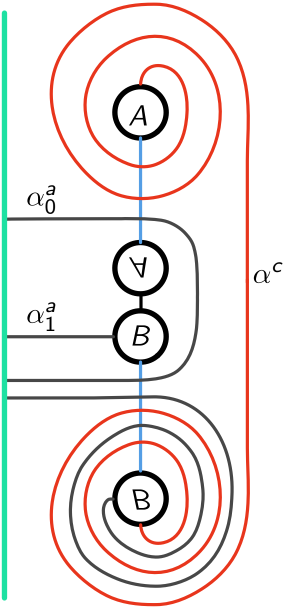

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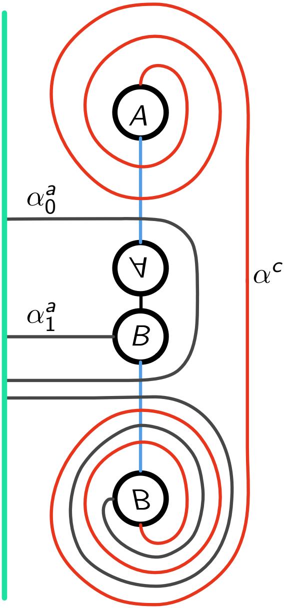
Order matters: $(M, \alpha_0^a, \alpha_1^a)$ and $(M, \alpha_1^a, \alpha_0^a)$ are different bordered manifolds.

So a bordered Heegaard diagram for M is a triple $(\mathcal{H}, \alpha_0^a, \alpha_1^a)$ where $\mathcal{H}$ is a Heegaard diagram of genus $g \geq 1$ with $g - 1$ α-curves.

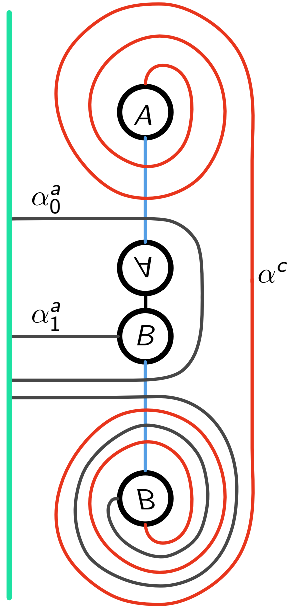
Bordered Heegaard Floer homology



To a Heegaard diagram $\mathcal{H}$, Heegaard Floer homology associates a chain complex $\widehat{\mathrm{CF}}(\mathcal{H})$ (over $\mathbb{F} = \mathbb{Z}/2$). $\widehat{\mathrm{HF}}(\mathcal{H})$ is independent of the choice of $\mathcal{H}$; write $\widehat{\mathrm{HF}}(Y)$.

This is due to Ozsváth and Szabó.

Bordered Heegaard Floer homology



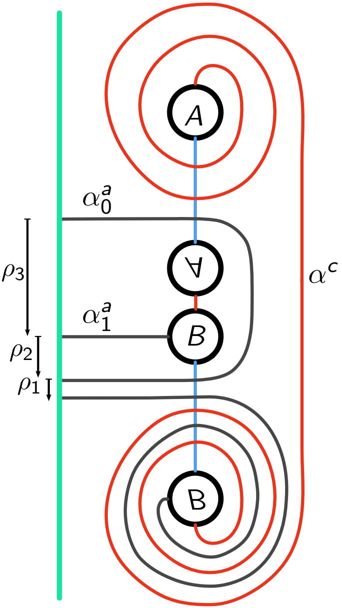
To a bordered Heegaard diagram $(\mathcal{H}, \alpha_0^a, \alpha_1^a)$, bordered Heegaard Floer homology associates a differential (graded) module

$$\widehat{\mathrm{CFD}}(\mathcal{H}, \alpha_0^a, \alpha_1^a)$$

over an algebra $\mathcal{A}$.
The homotopy type of this object is independent of the choice of $\mathcal{H}$ (but not the peripheral elements!); write $\widehat{\mathrm{CFD}}(M, \alpha_0^a, \alpha_1^a)$

This is due to Lipshitz, Ozsváth and Thurston.

The torus algebra



The algebra $\mathcal{A}$ is generated by two idempotents ι_0 and ι_1 and three Reeb elements ρ_1, ρ_2, ρ_3.

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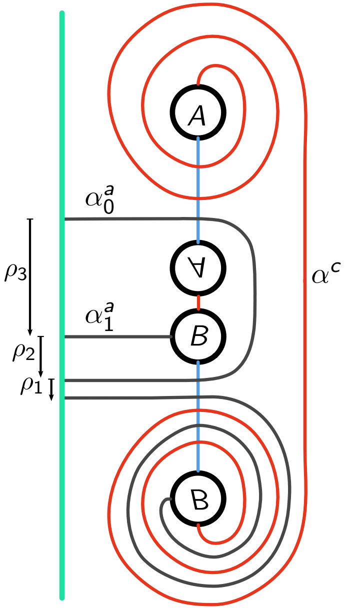

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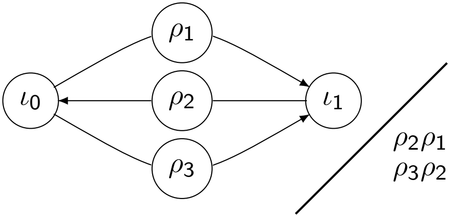
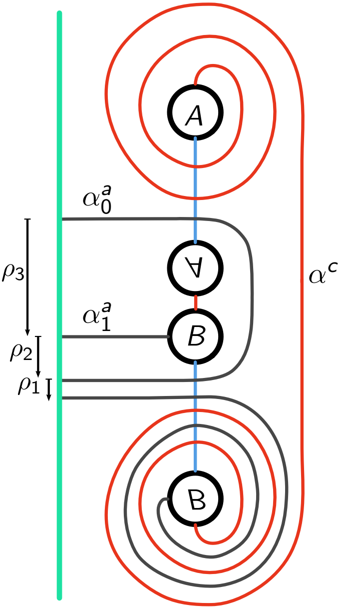
Multiplication may be described:



$$\rho_2\rho_1 = 0$$
$$\rho_3\rho_2 = 0$$

The torus algebra



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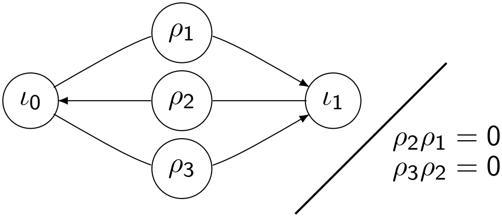
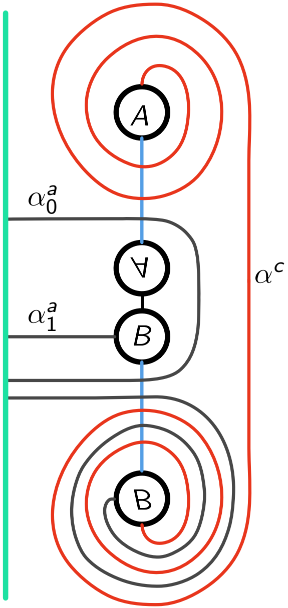


Write $\rho_2\rho_3 = \rho_{23}$, etc. so that $\mathcal{A}$ is 8 dimensional as a vector space over $\mathbb{F}$.

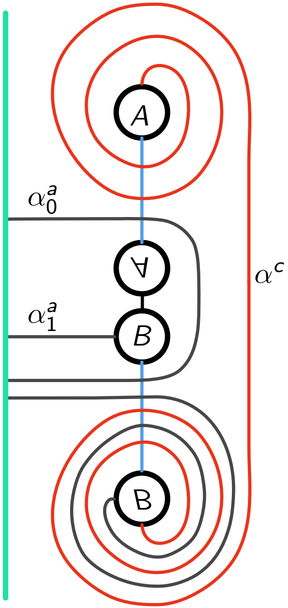
Differential modules over $\mathcal{A}$



In general, $\widehat{\text{CFD}}(M, \alpha_0^a, \alpha_1^a)$ is generated (as a vector space) by g-tuples of intersection points $\mathbf{x}$ between the collections $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

Exactly one of α_0^a or α_1^a will be occupied so that there is a splitting according to idempotents (depending on which of the α-arcs is occupied).

Differential modules over $\mathcal{A}$



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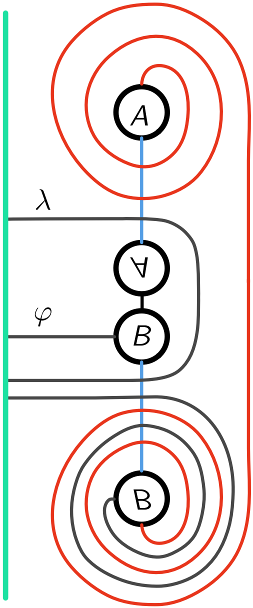
Note: as a differential module over $\mathcal{A}$,

$$\widehat{\mathrm{CFD}}(M, \alpha_0^a, \alpha_1^a)$$

has a differential $\partial(\mathbf{x}) = \sum_{\mathcal{A}} a \otimes \mathbf{y}$.

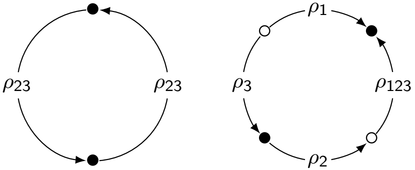
An example: the twisted I-bundle over the Klein bottle



For this example we have

$$\widehat{\mathrm{CFD}}(M, \alpha_0^a \simeq \lambda, \alpha_1^a \simeq \varphi)$$

described by the directed graph.



Example: There is a generator $\mathbf{x}$ in the ι_0-summand for which

$$\partial(\mathbf{x}) = \rho_1 \otimes \mathbf{u} + \rho_3 \otimes \mathbf{v}$$

where $\mathbf{u}, \mathbf{v}$ are generators in the ι_1-summand.

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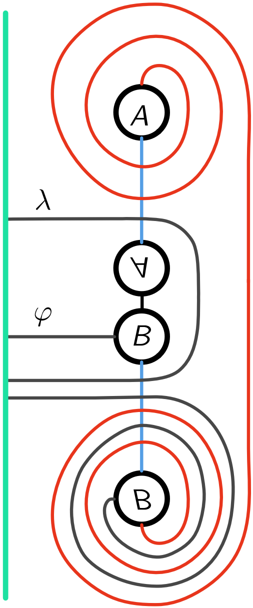

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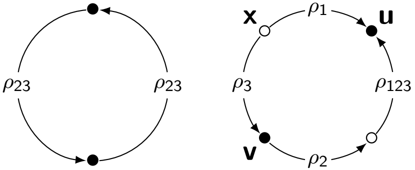

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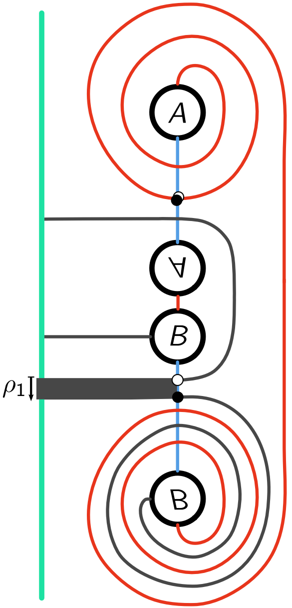

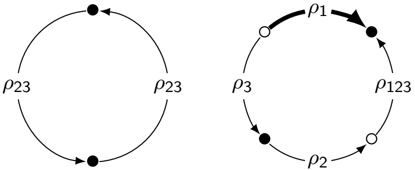

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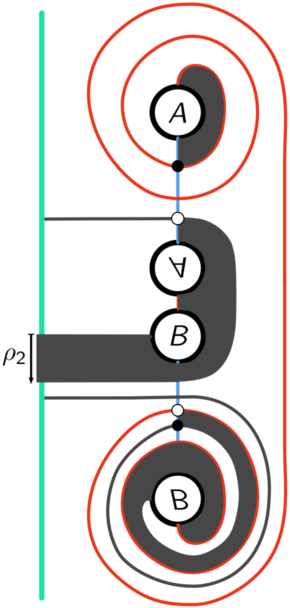

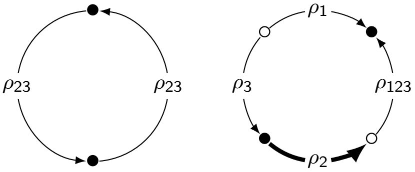

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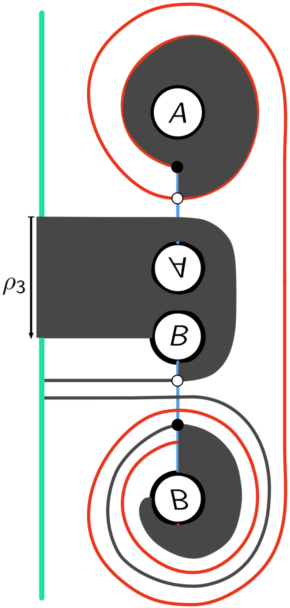
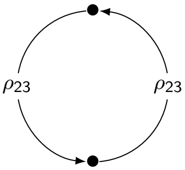
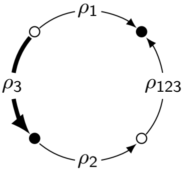






Simple objects in Heegaard Floer homology

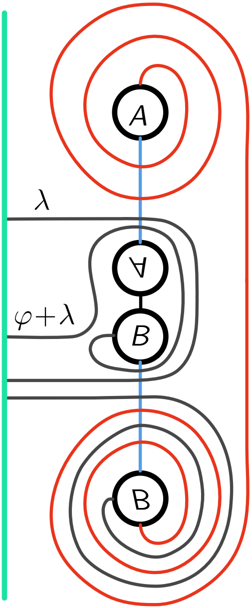
In general, for a $\mathbb{Q}$-homology sphere Y, $\operatorname{rk}\widehat{\mathrm{HF}}(Y) \geq |H_1(Y;\mathbb{Z})|$.
Equality is realized for lens spaces, and more generally we have:

Definition

A rational homology sphere Y is a Heegaard Floer homology lens space if $\operatorname{rk}\widehat{\mathrm{HF}}(Y) = |H_1(Y;\mathbb{Z})|$.

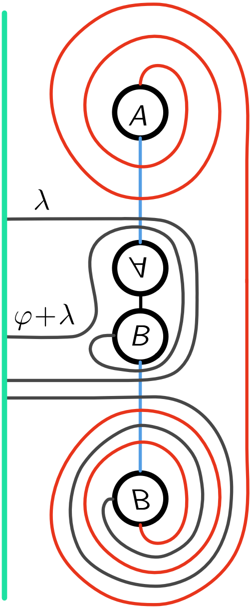
The term *Heegaard Floer homology lens space* has been shortened to *L-space* (for perhaps obvious reasons).

Twisting along the rational longitude



In general, bordered invariants are very sensitive to the choice of peripheral elements.

Twisting along the rational longitude



In general, bordered invariants are very sensitive to the choice of peripheral elements. However:

Proposition (Boyer-Gordon-W.)

$\widehat{\mathrm{CFD}}(M, \lambda, \varphi) \cong \widehat{\mathrm{CFD}}(M, \lambda, \varphi + n\lambda)$

This plays a central role in:

Theorem (Boyer-Gordon-W.)

If Y is a $\mathbb{Q}$-homology sphere admitting Sol geometry then Y is an L-space.

What is a simple object in bordered Floer theory?

Observe that (M, λ, φ) and $(M, \lambda, \varphi + n\lambda)$ are different bordered manifolds for each $n \in \mathbb{Z}$ (since $M \neq D^2 \times S^1$); the proposition may be interpreted as a *Heegaard Floer homology Alexander trick*.

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Definition

Let M be a $\mathbb{Q}$-homology solid torus with rational longitude λ. M is a Heegaard Floer homology solid torus if

$$\widehat{\mathrm{CFD}}(M, \lambda, \mu) \cong \widehat{\mathrm{CFD}}(M, \lambda, \mu + n\lambda)$$

for all $n \in \mathbb{Z}$, where $\langle \mu, \lambda \rangle \cong \pi_1(\partial M)$.

Examples of Heegaard Floer homology solid tori

The solid torus is a Heegaard Floer homology solid torus; the twisted I-bundle over the Klein bottle is a (non-trivial) Heegaard Floer homology solid torus.

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Theorem (W.)

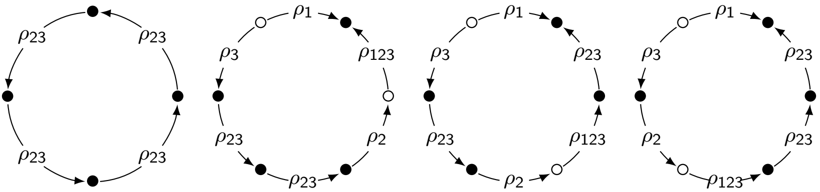
For each integer $n > 0$ there is a Heegaard Floer homology solid torus M_n which is a Seifert fibred space with

- $\pi_1(M_n) \cong \langle a, b | a^n b^n \rangle$,
- $H_1(M_n; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/n$,

and the Dehn filling along the rational longitude is $S^2 \times S^1$.

M_1 is the solid torus; M_2 is the twisted I-bundle over the Klein bottle.

The case $n = 4$: $\widehat{\mathrm{CFD}}(M_4, \lambda, \varphi)$



A construction

Given $Y = M \cup_h M'$ we have that

$$\widehat{\mathrm{CF}}(Y) \cong \widehat{\mathrm{CFA}}(M, \alpha_0, \alpha_1) \boxtimes \widehat{\mathrm{CFD}}(M', \alpha_0', \alpha_1')$$

where the bordered manifolds are chosen so that $h(\alpha_i') = \alpha_i$.

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Notice that if M' is a Heegaard Floer homology solid torus, we immediately get an infinite family of distinct 3-manifolds $\{Y_n\}$ with identical $\widehat{\mathrm{HF}}(Y_n)$: the homology in this setting only depends on the image of $\alpha'_0 = \lambda$ (as in Dehn surgery).

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Notice that if M' is a Heegaard Floer homology solid torus, we immediately get an infinite family of distinct 3-manifolds $\{Y_n\}$ with identical $\widehat{\mathrm{HF}}(Y_n)$: the homology in this setting only depends on the image of $\alpha'_0 = \lambda$ (as in Dehn surgery).

This is meant to justify the notion of a Heegaard Floer homology Alexander trick.

A final example: Dehn surgery revisited

Let M be the left-hand trefoil exterior with framing $\alpha = \lambda - 5\mu$, and let M_2 be the twisted I-bundle over the Klein bottle.

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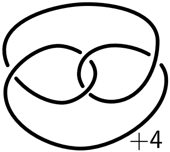
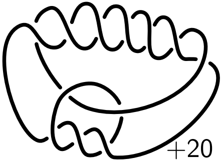


$(M, \mu, \alpha) \cup (M_2, \lambda, \varphi)$, that is,

$$\lambda \mapsto \mu.$$

An L-space ($\operatorname{rk} \widehat{\mathrm{HF}} = 20$):



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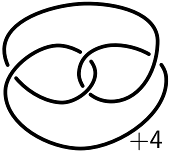

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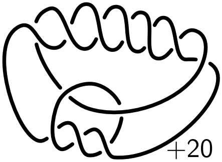

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Both examples give rise to infinite families, and both decompose into (nearly) the same bordered pieces.