Bordered Floer homology via immersed curves

Joint work with Jonathan Hanselman and Jake Rasmussen

Liam Watson
Université de Sherbrooke

MCA Montréal
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Heegaard Floer homology

Let M be a compact, connected, oriented three-manifold with torus boundary; fix a marked point $\star \in \partial M$.

Theorem (Hanselman-Rasmussen-W.)

The Heegaard Floer homology $\widehat{HF}(M)$ can be interpreted as a set of immersed curves in $T = \partial M \setminus \star$, up to regular homotopy.

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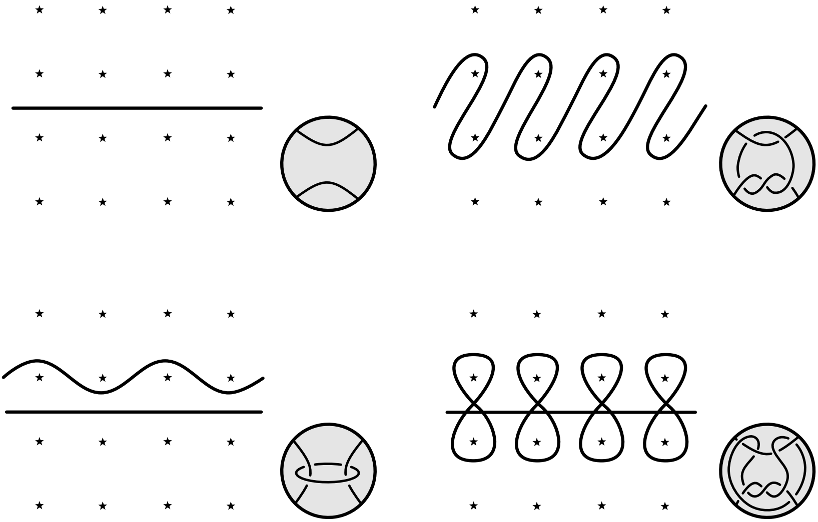
The Heegaard Floer homology $\widehat{HF}(M)$ can be interpreted as a set of immersed curves (possibly decorated with local systems) in $T = \partial M \setminus \star$, up to regular homotopy.

A local system is a finite dimensional vector space V (in our case, over $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$), together with an endomorphism $\Phi\colon V \to V$.

Some examples (in the cover $\mathbb{R}^2 \setminus \mathbb{Z}^2 \to T$)



The pairing theorem

Suppose further that $Y = M_0 \cup_h M_1$ where $h\colon \partial M_1 \to \partial M_0$ is an orientation reversing homeomorphism for which $h(\star_1) = h(\star_0)$.

Theorem (Hanselman-Rasmussen-W.)

$$\widehat{HF}(Y) \cong HF(\gamma_0, \gamma_1)$$

Here, $HF(\gamma_0, \gamma_1)$ computes the Lagrangian intersection Floer homology of

$$\gamma_0 = \widehat{HF}(M_0) \quad \text{and} \quad \gamma_1 = h^!\big(\widehat{HF}(M_1)\big)$$

in the punctured torus $T = \partial M_0 \setminus \star_0$. The function $h^!$ composes h with the hyperelliptic involution on T.

Example: splicing right-handed trefoils

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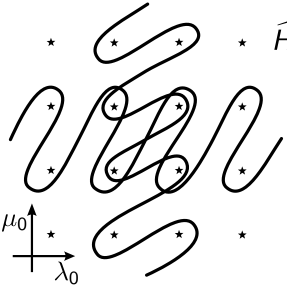

$$\widehat{HF}(M_0 \cup_h M_1) \cong \mathbb{F}^7$$

$$h(\mu_1) = \lambda_0$$
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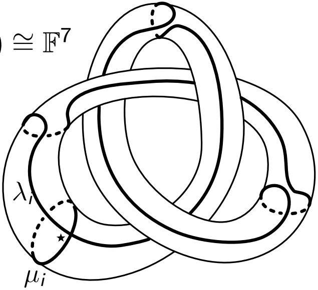

See Hedden-Levine to compare this with a direct bordered Floer calculation of this particular splice.

Application: The L-space gluing theorem

Definition

A rational homology sphere Y is an L-space whenever

$$\dim \widehat{HF}(Y) = |H_1(Y; \mathbb{Z})|$$

Question

When is $M_0 \cup_h M_1$ an L-space?

When one (or both) of the M_i is a solid torus, the answer is "sometimes". Define

$$\mathcal{L}_M = \{\alpha \,|\, \text{the Dehn filling } M(\alpha) \text{ is an L-space}\} \subset \mathcal{S}_M$$

Application: The L-space gluing theorem

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Theorem (Hansleman-Rasmussen-W.)

Suppose M_i is irreducible and boundary irreducible (in particular, $M_i \ncong D^2 \times S^1$). Then $M_0 \cup_h M_1$ is an L-space if and only if

$$\mathcal{L}^\circ_{M_0} \cup h(\mathcal{L}^\circ_{M_1}) = \mathcal{S}_{M_0}$$

Special cases of this were known: Hedden-Levine, Hanselman, Hanselman-W., Hanselman-Rasmussen-Rasmussen-W.

Consequences of the L-space gluing theorem

Corollary

Set $N = |H_1(Y; \mathbb{Z})|$ *for* $Y = M_0 \cup_h M_1$ *with* $M_i \neq D^2 \times S^1$*. If* $N = 1, 2, 3, 6$ *then* Y *is not and L-space.*

In particular, there do not exist toroidal integer homology sphere L-spaces (see also Eftekhary).

Conjecture (Ozsváth-Szabó)

The only prime integer homology sphere L-spaces are the three-sphere and the Poincaré homology sphere.

Consequences of the L-space gluing theorem

A knot in the three-sphere admitting non-trivial L-space surgeries is called an L-space knot. That is, such K are characterized by the property $|\mathcal{L}_{S^3\setminus\nu(K)}| > 1$.

Corollary

Suppose K is a satellite L-space knot. Then both the pattern knot and the companion knot must be L-space knots.

This was conjectured by Hom-Lidman-Vafaee. More can be said about the companion knot; see Baker-Motegi.

Application: Degree one maps

Question

Given a degree one map $Y \to Y_0$, what is the relationship between $\widehat{HF}(Y)$ and $\widehat{HF}(Y_0)$?

Given an integer homology sphere $M_0 \cup_h M_1$, consider the slope

$$\alpha_h = h(\lambda_1) \in \mathcal{S}_{M_0}$$

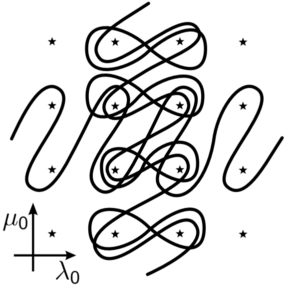
Note that this is a meridian for M_0, that is, $\Delta(\lambda_0, \alpha_h) = 1$.

Theorem (Hanselman-Rasmussen-W.)

Let $Y = M_0 \cup_h M_1$ and $Y_0 = M_0(\alpha_h)$. Then there is a degree one map $Y \to Y_0$ and

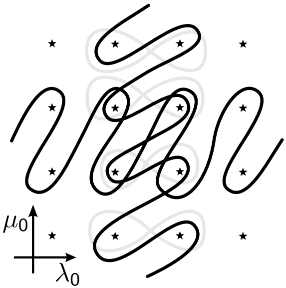
$$\dim \widehat{HF}(Y) \geq \dim \widehat{HF}(Y_0)$$

A sketch of the proof



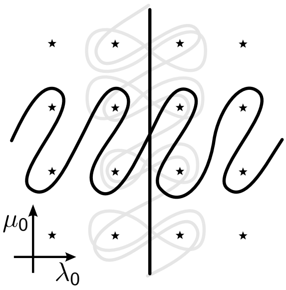
Step 0: make any local systems appearing trivial

A sketch of the proof



Step 0: make any local systems appearing trivial
Step 1: remove any closed components

A sketch of the proof



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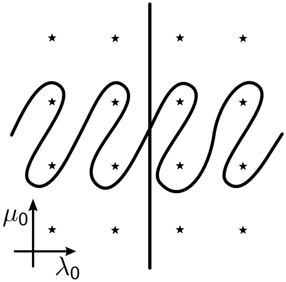
A sketch of the proof



Step 0: make any local systems appearing trivial
Step 1: remove any closed components
Step 2: pull the remaining curve tight
Step 3: check that none of these steps created new intersection points

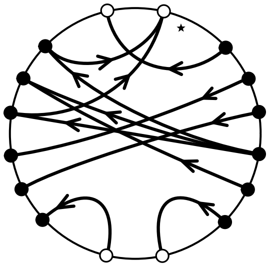
Where does $\widehat{HF}(M)$ come from?

The curve-set $\widehat{HF}(M)$ is a geometric interpretation of the bordered Floer homology $\widehat{\mathrm{CFD}}(M, \alpha, \beta)$, which was defined by Lipshitz-Ozsváth-Thurston.

The invariant $\widehat{\mathrm{CFD}}(M, \alpha, \beta)$ is a type D structure, which is a linear-algebraic object defined over an algebra $\mathcal{A}$ associated with $T = \partial M \setminus \star$.

The interpretation $\widehat{HF}(M)$ comes from describing type D structures as geometric objects in T, and then providing a structure theorem that simplifies them.

Type D structures associated with a point



Consider a 0-handle with a marked point $\star$ near the boundary.

Consider points (stations) on the boundary collected into groups (towns).

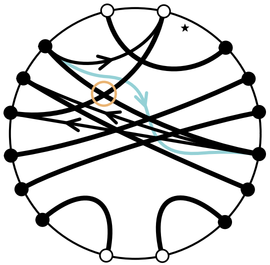
A type D structure is a train track that

(1) only travels to a *next* town;

(2) doesn't pass the basepoint; and

(3) has an even number of possible connections.

Type D structures associated with a point

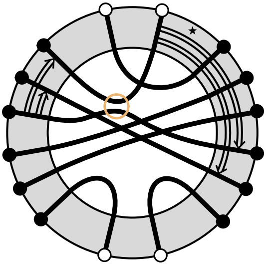


Desired property: Extendability

An extension of a type D structure is a rail system upgrade.

The additional tracks can pass $\star$ at most once.

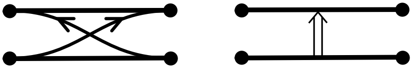
Type D structures associated with a point



Lemma

Any extension of a type D structure is equivalent to one in standard form: a collection of properly embedded arcs, together with crossover arrows running clockwise along the boundary.

Convention:



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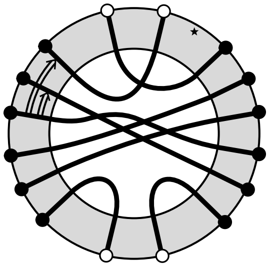
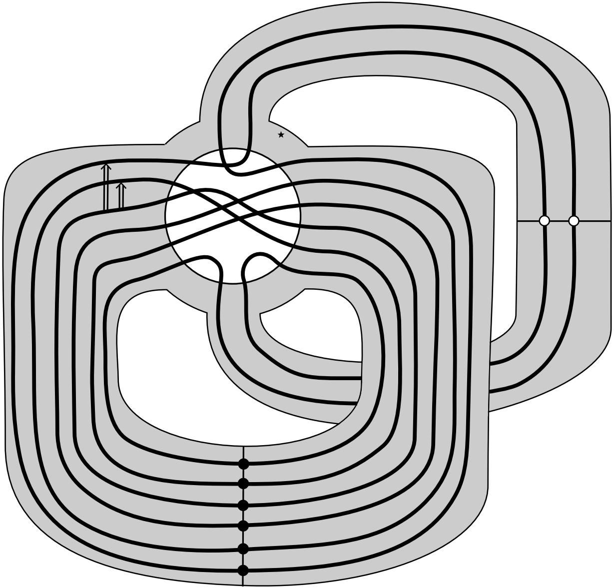


Crossover arrows moving between groups can be removed.

Structure Theorem

Every extendable type D structure associated with a zero handle can be put in the (simplified) standard form illustrated on the left.

Adding 1-handles



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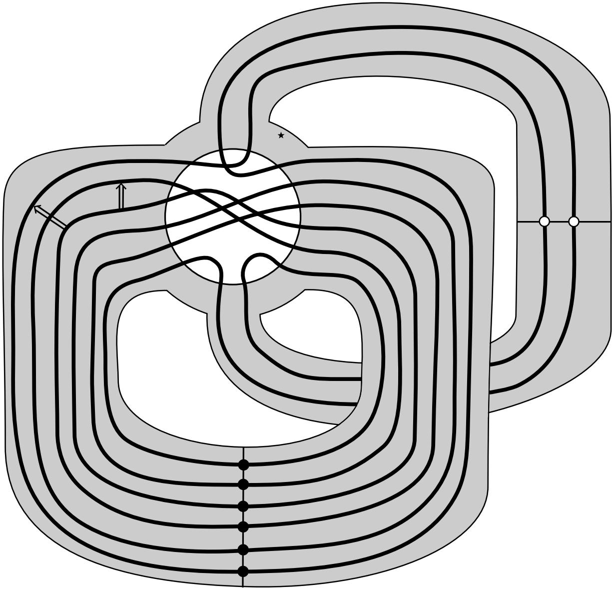

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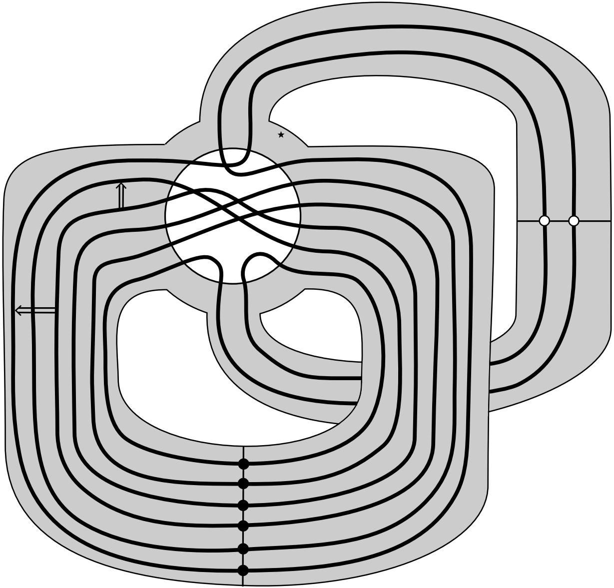

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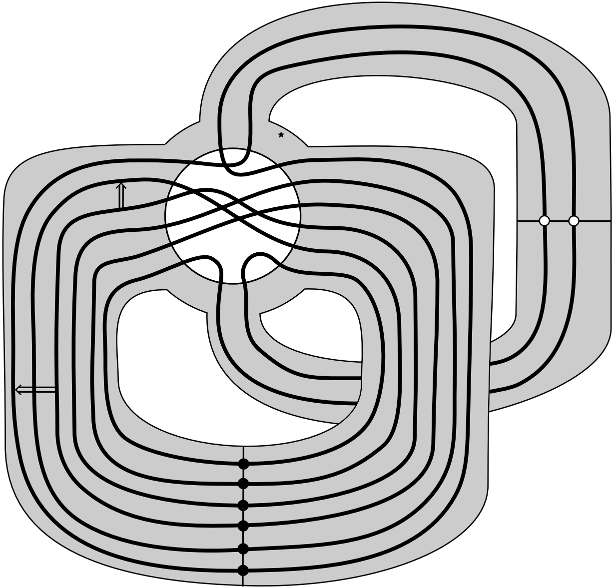

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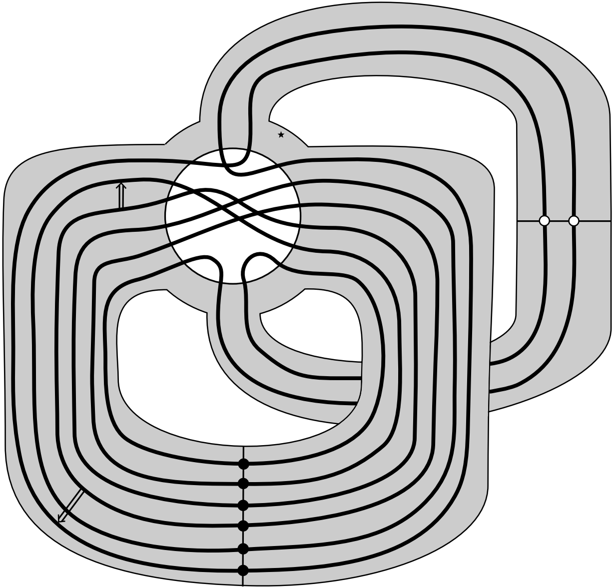

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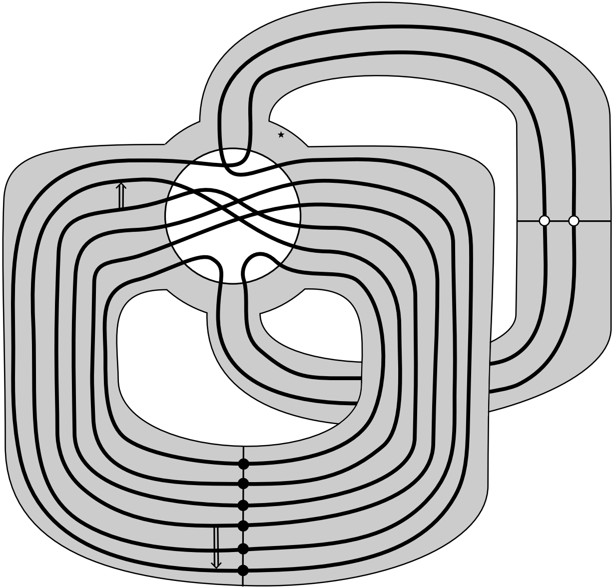

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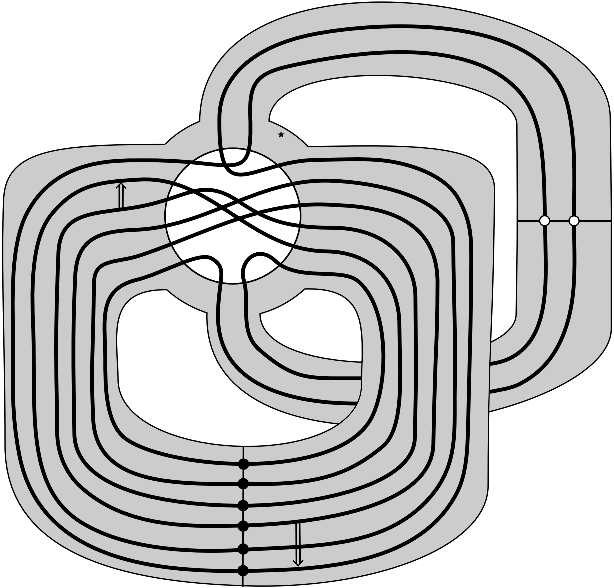

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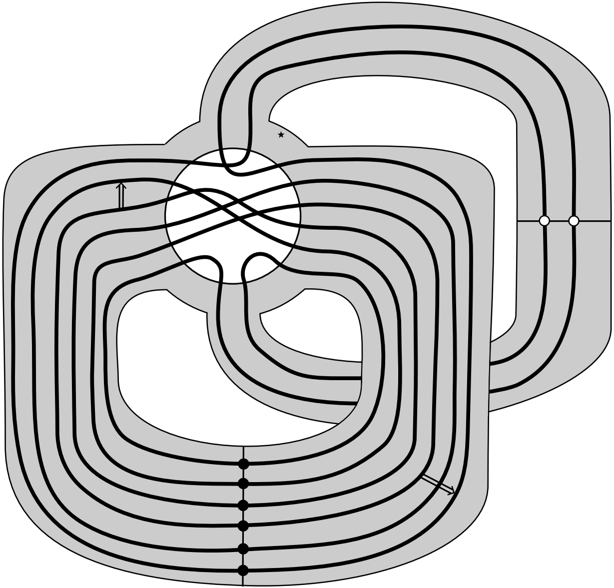

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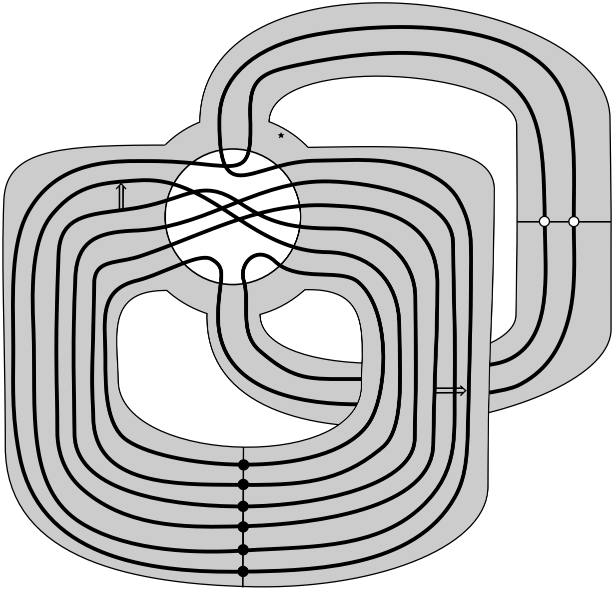

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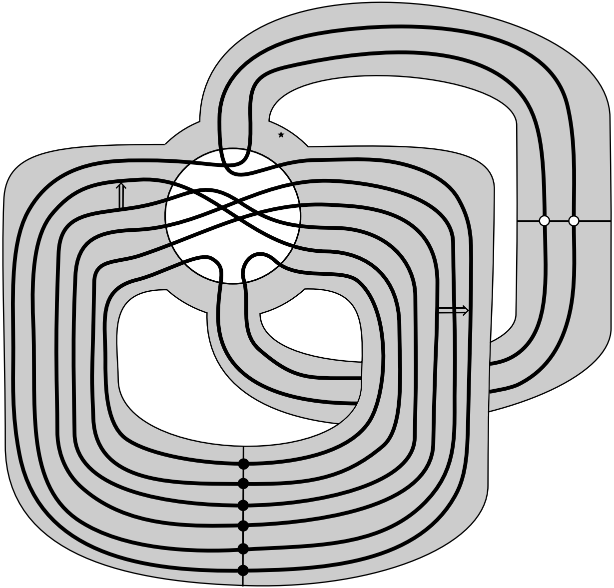

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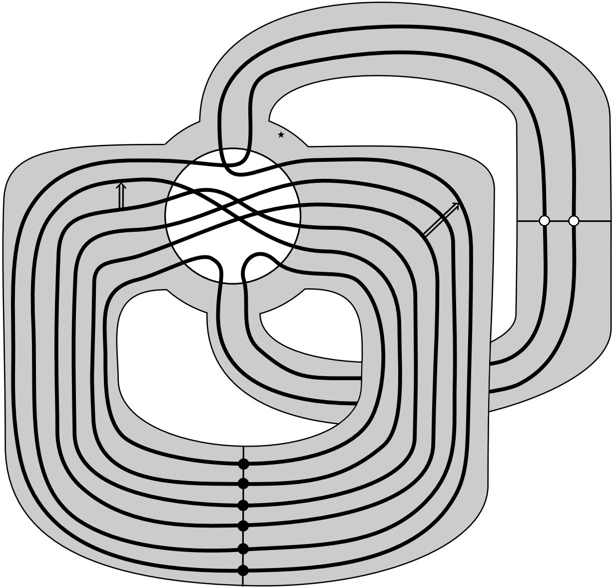

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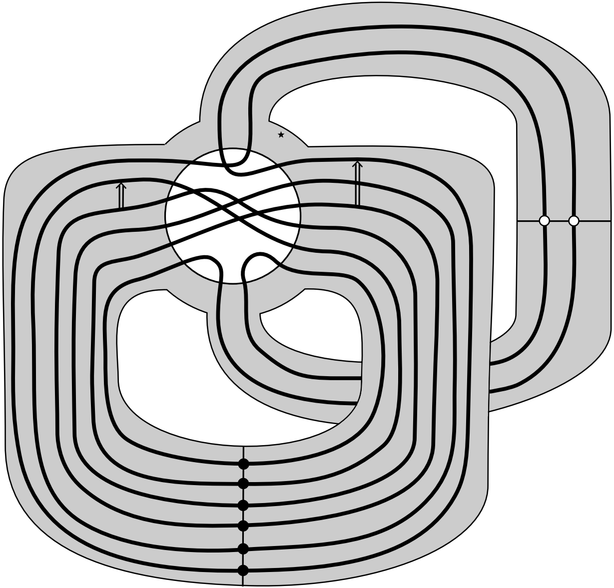

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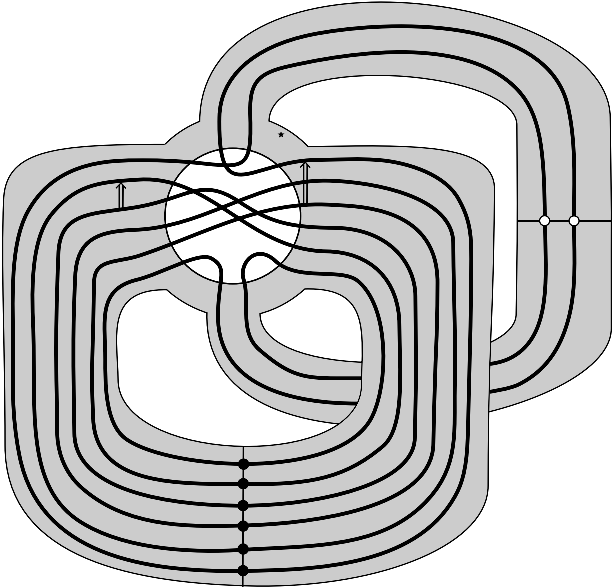

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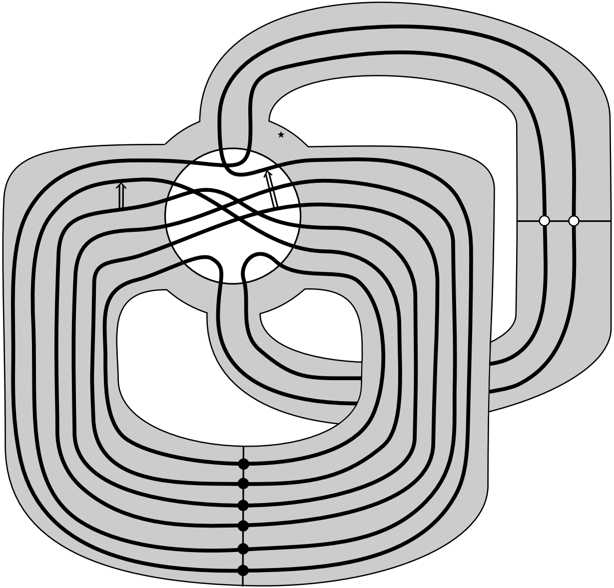

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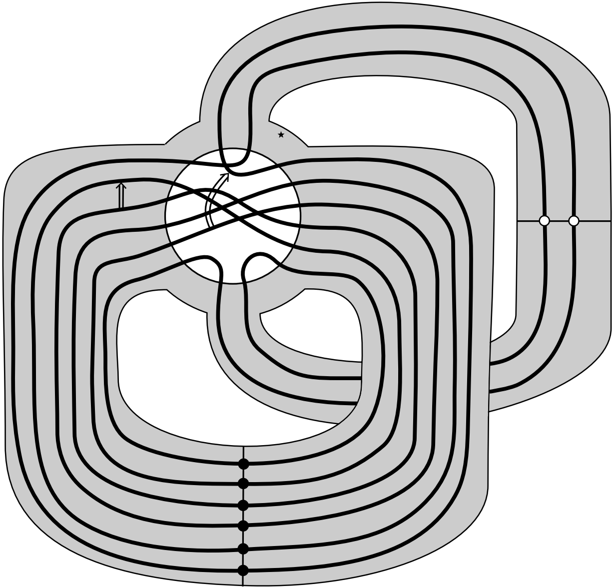

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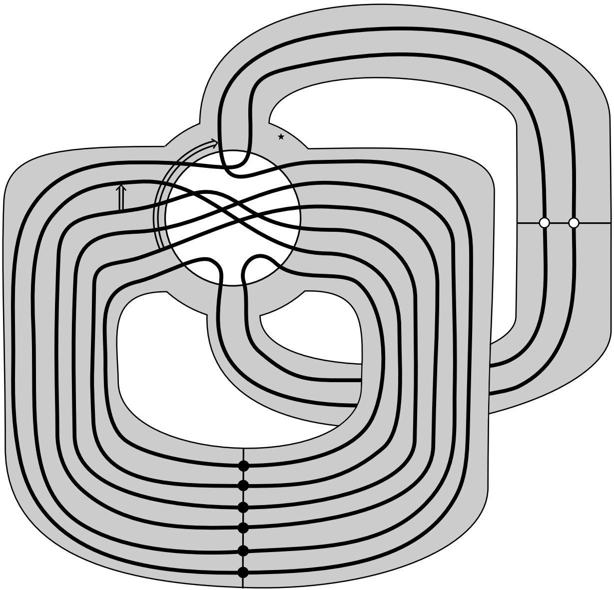

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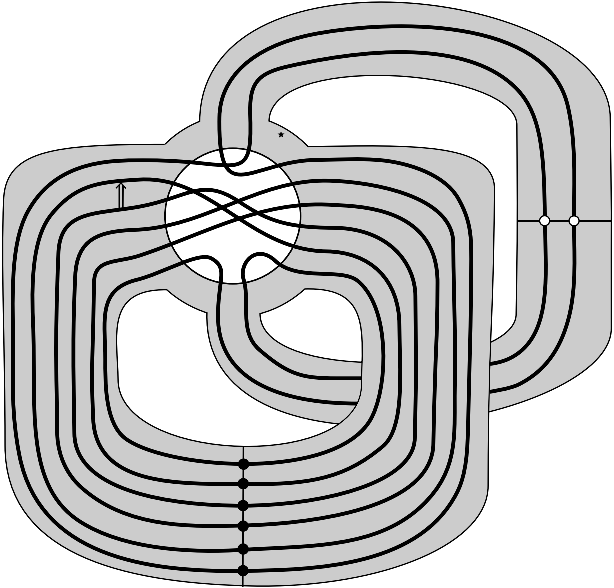

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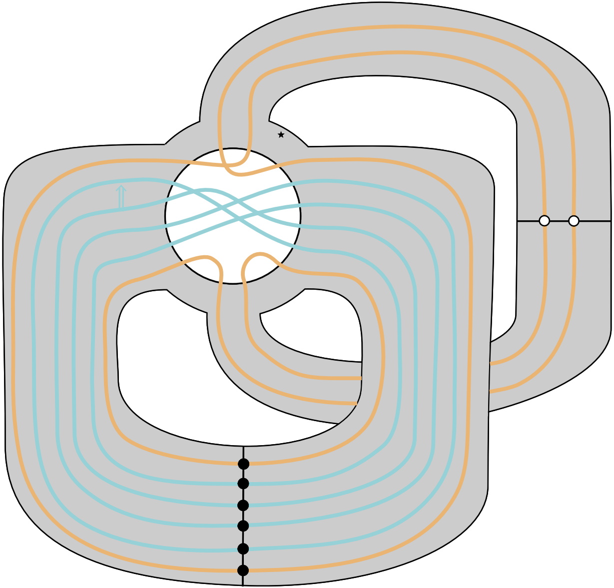

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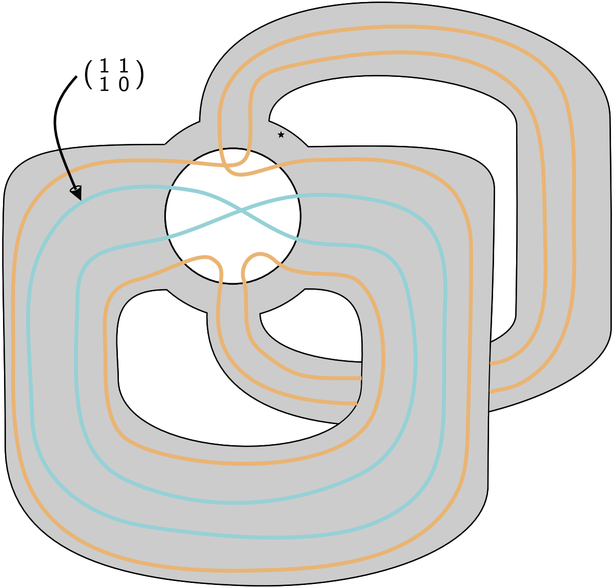

Three-manifold invariants

Structure Theorem

Extendable Type D structures in a surface with a fixed 0*- and* 1*-handle decomposition are immersed curves with local systems.*

Extension Theorem

The type D structure $\widehat{\mathsf{CFD}}(M, \alpha, \beta)$ *is extendable.*