Quantum Unique Ergodicity

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Scarring

[Heller 1984]

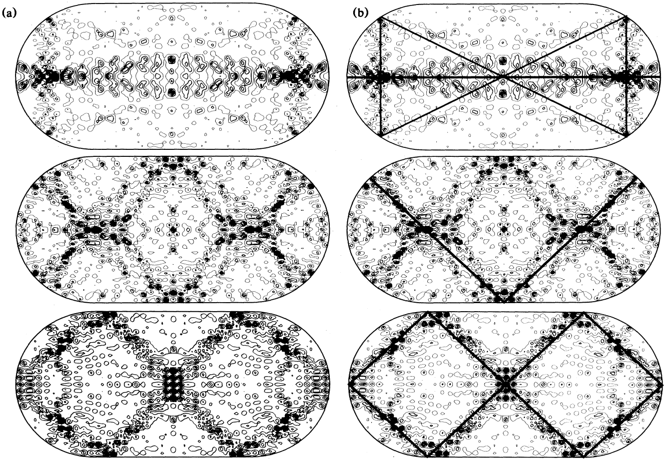
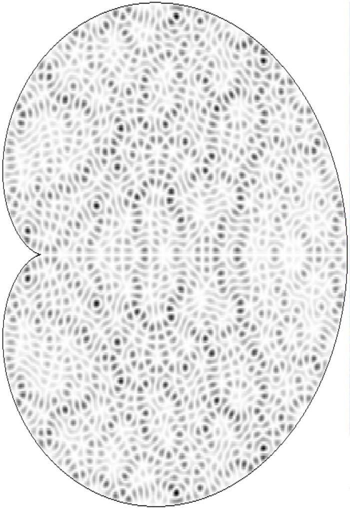
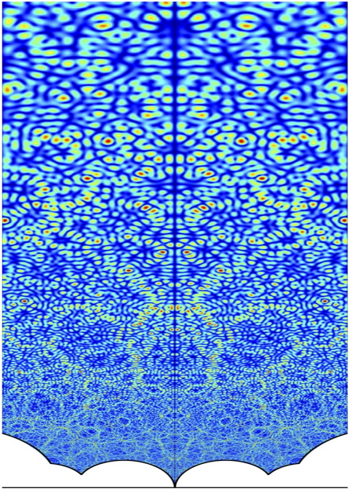


FIG. 2. Left column, three scarred states of the stadium; right column, the isolated, unstable periodic orbits corresponding to the scars.

Other examples





(Images: Bäcker, Stromberg)

Quantum Unique Ergodicity

Problem: What happens as $\lambda \to \infty$? What is a "feature"?

Pointwise How big does $\|u_\lambda\|_\infty$ get as $\lambda \to \infty$?

Weakly What happens to $\int |u_\lambda|^2 f \, \mathrm{dvol}$ as $\lambda \to \infty$?

Theorem (Schnirel'man–Zelditch–Colin de Verdière)

If the billiard dynamics is chaotic (ergodic) then for almost all eigenfunctions $\int |u_\lambda|^2 f \, \mathrm{dvol} \to \frac{1}{\mathrm{vol}} \int f \, \mathrm{dvol}$

Conjecture (Rudnick–Sarnak)

On a manifold of negative sectional curvature, replace "almost all" with "all".

Hassell 2008: For stadium billiard, can't remove "almost".

Plan

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1. Bounds on eigenfunctions on the tree and in the plane
2. “Classical” and “quantum” mechanics
3. “Arithmetic” QUE
4. Without arithmetic
5. Negative results for approximate eigenfunctions

A pointwise bound

Theorem (Hörmander bound)

$\|u_\lambda\|_\infty \leq C\lambda^{\frac{n-1}{4}} \|u_\lambda\|_2.$

Proof (in spirit).

Use u_λ as the initial condition for an evolution equation, e.g.

$$i\hbar\frac{\partial}{\partial t}\psi(t,x) = -\Delta_x\psi(t,x).$$

- $\psi(t,x) = e^{-i\lambda t}u_\lambda(x)$ is a solution.
- But solutions tend to follow classical trajectories.
- So $\psi(t,x)$ looks like u_λ "averaged" over a region near x, and can relate $\psi(t,x)$ to $\|u_\lambda\|_2$.

□

Some physics

The Space of Lattices

Move to *curved geometry* and *periodic boundary conditions.*

- $\mathcal{P}_n = \{\text{symmetric, positive-definite } n\text{-matrices } X, \det(X) = 1\}$
- $\mathrm{SL}_n(\mathbb{R})$ acts by $g \cdot X := gXg^t$, preserving metric: $\operatorname{dist}(\mathrm{Id}, X) = \left(\sum_{i=1}^n |\log \mu_i|^2\right)^{1/2}$, $\mu_i = \text{eigenvalues}$.
- For $n = 2$, $\mathcal{P}_n$ is the hyperbolic plane.
- Study the quotient $\mathcal{L}_n = \mathrm{SL}_n(\mathbb{Z})\backslash\mathcal{P}_n$ $=$ isometry classes of unimodular lattices in $\mathbb{R}^n$.

Arithmetic QUE

- Domain has *number-theoretic symmetries*, manifest as *Hecke operators* $(T_p f = \sum_{y \sim x} f(y))$

$$T_p \Delta = \Delta T_p, \qquad T_p T_q = T_q T_p$$

- Study limits of joint eigenfunctions. Start with $n = 2$:
- Rudnick–Sarnak 1994: limits don't scar on closed geodesics.
- Iwaniec–Sarnak 1995: savings on Hörmander bound
  - small balls have small mass
- Bourgain–Lindenstrauss 2003: limits have positive entropy
  - small dynamical balls have small mass
- Lindenstrauss 2006: from this get equidistribution.

Higher-rank QUE

- What *about* $n \geq 3$*?*
- No longer negatively curved – extend Rudnick–Sarnak conjecture
- S–Venkatesh 2007: limits respect *Weyl chamber flow*
- S–Venkatesh: (non-degenerate) limits are uniformly distributed if n is prime (division algebra quotient).

QUE Results proceed by

- Lift to the bundle where classical flow lives.
- Bound mass of dynamical balls ("positive entropy")
- Apply measure-classification results to identify the limit.

QUE on general manifolds

- In $\mu(B(C,\varepsilon)) \ll \varepsilon^h$, h measures the *complexity of* μ.
- Related to the *metric entropy* $h(\mu)$.
- Anantharaman ∼2003: On a manifold of negative curvature, every quantum limit has positive entropy.
- Anatharaman + others: quantitative improvements
- Idea: “quantum partition”

Applied to the space of lattices

- $\mathcal{L}_n$ not negatively curved (has flats).
- Nevertheless limits have positive entropy:
  - Microlocal calculus adapted to locally symmetric spaces.
  - Entropy contribution from "rapidly expanding" directions.
- Measure-classification
  - Restriction on possible ergodic components.
  - Use *quantitative* entropy bound.

Theorem (Anantharaman–S)

Let $X = \Gamma \backslash \mathcal{P}_3$ be compact. Then every quantum limit on X is at least $\frac{1}{4}$ Haar measure.

New uncertainty principle

- Density is now known for $n = 2$:

Theorem (Dyatlov–Jin 2018)

Every quantum limit on a compact hyperbolic surface has full support.

Theorem (Dyatlov–Jin–Nonnenmacher 2019)

The same on a compact surface with Anosov geodesic flow.

Approximate eigenfunctions

- Method of Anantharaman applies to *approximate eigenfunctions.*

$$\|\Delta u_\lambda + \lambda u_\lambda\| \leq C\frac{\sqrt{\lambda}}{\log \lambda}$$

- Entropy depends on C.

Problem

What are the possible limits of these “log-scale quasimodes”?

Scarring of quasimodes

Problem

On a manifold M, construct log-scale quasimodes which concentrate on singular measures

$$\|\Delta u_\lambda + \lambda u_\lambda\| \leq C \frac{\sqrt{\lambda}}{\log \lambda}$$

$$\lim_{\lambda\to\infty} \int |u_\lambda|^2 f \,\mathrm{dvol} = \int f \,\mathrm{d}\mu$$

- Brooks 2015: $M = \text{hyperbolic surface}$, $\mu = \text{geodesic}$.
  - Uses the geometry explicitely (Eisenstein packets)
- Eswarathasan–Nonnenmacher 2016: M=any surface, $\mu = \text{hyperbolic geodesic}$.

High dimensions

Theorem (Eswarathasan–S 2017)

Let M be a hyperbolic manifold, and let $N \subset M$ be a compact totally geodesic submanifold. Then there is a sequence of log-scale quasimodes uniformly concentrating on N.

- Includes the case $N = \text{closed geodesic}$.
- Actually, any quantum limit on N achievable.

Corollary

(M compact) every invariant measure on M is a limit of log-scale quasimodes.

Proof.

In a hyperbolic system, closed orbits are dense in the space of invariant measures. □