Factors of Gibbs measures on subshifts

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UBC Mathematics

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Acknowledgments

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I am grateful to be supervised by Lior, Brian Marcus, and Omer Angel.

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We are very grateful to Brian for his generous support and supervision, and to Tom Meyerovitch for his generous advice throughout this work.

Goals for this presentation

In this talk, I hope to communicate to you:

- Roughly two definitions of a Gibbs measure on a subshift and why they are equivalent
- A property defining a class of factor maps that preserve Gibbsianness, and some elements of the proof
- A Lanford-Ruelle theorem for irreducible sofic shifts on $\mathbb{Z}$

On Thursday, we can go into more detail, as interest dictates

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- Sofic shift: factor of an SFT
- All measures G-invariant Borel probability measures

Finite thermodynamics

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What about infinite volume?

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$$H_\Lambda^\Phi(x) = \sum_{\substack{\Delta \Subset G \\ \Delta \cap \Lambda \neq \emptyset}} \Phi_\Delta(x)$$

This converges when Φ is *absolutely summable*

$$\|\Phi\| = \sum_{\substack{\Lambda \Subset G \\ e \in \Lambda}} \|\Phi_\Lambda\|_\infty < \infty$$

Potentials

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Regularity: if G has polynomial growth $|B_n| \sim n^d$, define

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We called this the shell norm, vs. the volume norm

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where $a_\Lambda \geq 0$ are weights with $\sum_{g \in G} a_{g^{-1}\Lambda} = 1$.

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A potential f with finite volume norm induces an interaction Φ^f (with $f = A_{\Phi^f}$) by a telescoping construction due to Muir, building on Ruelle.

The Gibbs relation

Let $(\Lambda_N)_{N=1}^{\infty}$ be a sequence of finite sets exhausting G, and define relations $\mathfrak{T}_{X,N} \subset X^2$ by

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Let $\mathfrak{T}_X = \bigcup_{N=1}^{\infty} \mathfrak{T}_{X,N}$ (tail/asymptotic/Gibbs relation)
Equivalently, for all $x \in X$,

$$(x, x') \in \mathfrak{T}_X \iff \lim_{g \to \infty} d(x \cdot g, x' \cdot g) = 0$$

Cocycles

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If $\|\Phi\| < \infty$ and $\mathrm{diam}(\Lambda)^d / |\Lambda| \leq C$ then these agree,

$$\phi_\Phi = \phi_{A_\Phi}$$

The DLR equations

Definition

For a measure μ, a cocycle ϕ, a finite $\Lambda \Subset G$, and a Borel $A \subseteq X$, the Dobrushin-Lanford-Ruelle equation reads

$$\mu(A \,|\, \mathcal{F}_{\Lambda^c})(x) = \sum_{\eta \in \mathcal{A}^\Lambda} \left[\sum_{\zeta \in \mathcal{A}^\Lambda} \exp(\phi(\eta x_{\Lambda^c}, \zeta x_{\Lambda^c})) \mathbf{1}_X(\zeta x_{\Lambda^c}) \right]^{-1} \mathbf{1}_A(\eta x_{\Lambda^c})$$

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Examples (with $\Phi \equiv 0$)

- yes: Parry measure on irreducible edge shift (uniform on paths of length n between two states)
- no: point mass on sunny-side-up shift (the measure doesn't know about the yolk)

Conformal measures

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Requires nonsingularity: $\mu(A) = 0 \implies \mu(\mathfrak{T}_X(A)) = 0$

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Going forward, we'll use the term Gibbs measure

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Problem: find sufficient topological conditions on X such that Gibbs $\iff$ equilibrium

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Theorem (Dobrushin, 1969; formulation due to Ruelle)

If $X \subseteq \mathcal{A}^{\mathbb{Z}^d}$ satisfies condition (D) and $\|\Phi\| < \infty$, then any Gibbs measure on X for Φ is an equilibrium measure for A_Φ.

Topological Markov properties

- A subshift $X \subseteq \mathcal{A}^G$ has the *topological Markov property* (TMP) if $x_{\Lambda_1} x'_{\Lambda_2^c} \in X$ whenever x, x' agree on $\Lambda_2 \setminus \Lambda_1$ for Λ_2 large enough depending on Λ_1

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- X has the *strong TMP* if we can take $\Lambda_2 = \Delta\Lambda_1$ for a fixed finite $\Delta \Subset G$

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Theorem (2020)

If $X \subset \mathcal{A}^G$ *is irreducible and has the TMP, and* π *essentially respects* $\mathfrak{T}_X$*, then* μ *fully supported ergodic Gibbs for* $\pi^*\phi$ $\implies$ $\pi_*\mu$ *Gibbs for* ϕ*.*

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Meyerovitch (2013) presents non-sofic examples with Lanford-Ruelle-like properties (equilibrium $\implies$ Gibbs)

- Skew products of Kalikow type (T-T^{-1})
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For $Y \subseteq \mathcal{A}^{\mathbb{Z}}$ an irreducible sofic shift and $f \in \mathrm{SV}_d(X)$, every equilibrium measure for f is Gibbs for f.

Respecting the Gibbs relation

- If $X \subseteq \mathcal{A}^G$ is irreducible and has the TMP, and $\pi : X \to Y$ essentially respects $\mathfrak{T}_X$, then π satisfies a weak almost invertibility property (doesn't seem to imply that (X, μ) and $(Y, \pi_*\mu)$ are measurably conjugate)

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Theorem (2020)

Let $X \subseteq \mathcal{A}^{\mathbb{Z}}$ be a mixing SFT, $\pi : X \to Y$ a finite-to-one factor code, and $f \in \mathrm{SV}(Y)$. If μ is a Gibbs measure for $\pi^* f$ then $\pi_* \mu$ is a Gibbs measure for f.

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- Hypotheses required to show that in almost every point, every finite pattern appears infinitely often
  - If X is strongly irreducible then every Gibbs measure on X has full support

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Possible discussion topics

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Feel free to reach out before Thursday afternoon!

▷ sophmac at math dot ubc dot ca

On Thursday we can discuss any questions or comments I have received, and see where the discussion goes. If it seems appropriate, I can take a poll, like Lior did last week, on prepared selections from the list above.

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