Arithmetic and geometric properties of self-similar sets

Pablo Shmerkin

Torcuato Di Tella University
and CONICET

Pacific Dynamics Seminar, 4 June 2020

Self-similar sets

Definition

A compact set $E \subset \mathbb{R}^d$ is self-similar if there exist similarities $(f_i(x) = r_i O_i x + t_i)_{i=1}^m$ with $0 < r_i < 1$, $O_i \in \mathbb{O}_d$, $t_i \in \mathbb{R}^d$ such that

$$E = \bigcup_{i=1}^m f_i(E).$$

- If $r_i \equiv r$ and $O_i \equiv O$ we say that E is a homogeneous self-similar set.
- In $\mathbb{R}$, $O_i(x) = x$ or $-x$ and in $\mathbb{R}^2$, $O_i(x) = R_{\theta_i}(x)$ (possibly composed with a reflection).

Self-similar sets

Definition

A compact set $E \subset \mathbb{R}^d$ is self-similar if there exist similarities $(f_i(x) = r_i O_i x + t_i)_{i=1}^m$ with $0 < r_i < 1$, $O_i \in \mathbb{O}_d$, $t_i \in \mathbb{R}^d$ such that

$$E = \bigcup_{i=1}^{m} f_i(E).$$

- If $r_i \equiv r$ and $O_i \equiv O$ we say that E is a homogeneous self-similar set.
- In $\mathbb{R}$, $O_i(x) = x$ or $-x$ and in $\mathbb{R}^2$, $O_i(x) = R_{\theta_i}(x)$ (possibly composed with a reflection).

Self-similar sets

Definition

A compact set $E \subset \mathbb{R}^d$ is self-similar if there exist similarities $(f_i(x) = r_i O_i x + t_i)_{i=1}^m$ with $0 < r_i < 1$, $O_i \in \mathbb{O}_d$, $t_i \in \mathbb{R}^d$ such that

$$E = \bigcup_{i=1}^m f_i(E).$$

- If $r_i \equiv r$ and $O_i \equiv O$ we say that E is a homogeneous self-similar set.
- In $\mathbb{R}$, $O_i(x) = x$ or $-x$ and in $\mathbb{R}^2$, $O_i(x) = R_{\theta_i}(x)$ (possibly composed with a reflection).

Some homogeneous self-similar sets on the line

Figure: The middle-thirds Cantor set (points whose base 3 expansion has digits 0 and 2)

Some homogeneous self-similar sets on the line

Figure: The middle-one quarter Cantor set (points whose base 4 expansion has digits 0 and 3)

Some homogeneous self-similar sets on the line

Figure: A self-similar set with overlaps

Some planar self-similar sets

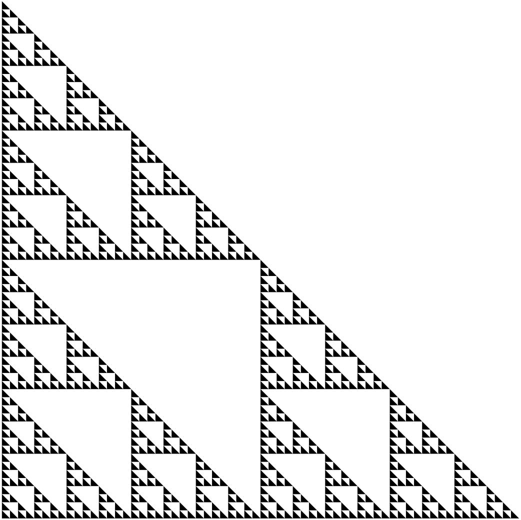


Figure: The Sierpiński triangle

Some planar self-similar sets

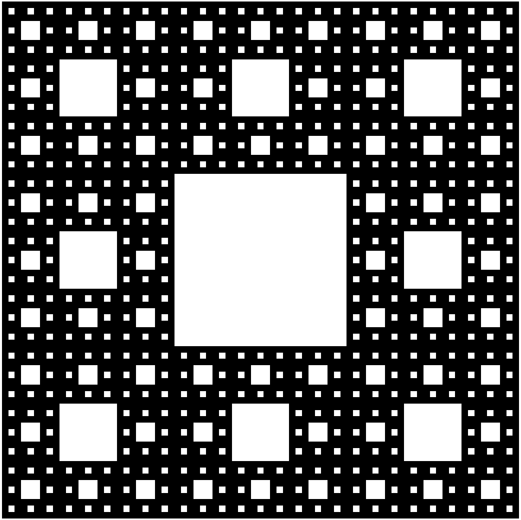


Figure: The Sierpiński carpet

Some planar self-similar sets

Figure: The one-dimensional Sierpiński gasket

Some planar self-similar sets

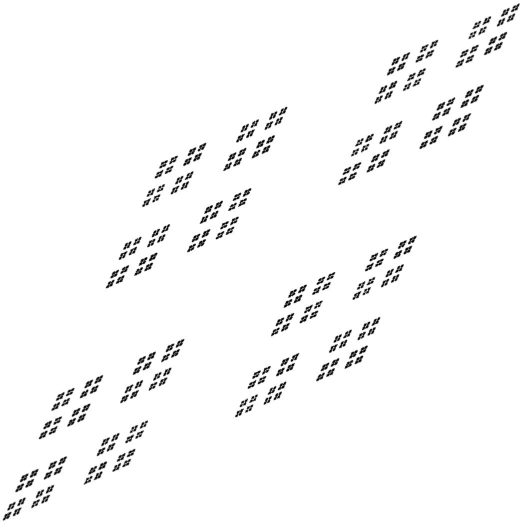


Figure: A non-carpet, no-rotations self-similar set

Some planar self-similar sets

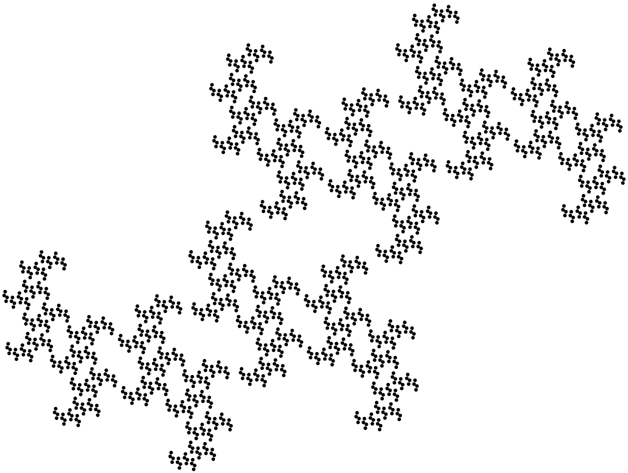


Figure: A complex Bernoulli convolution (two maps, rotation)

Some planar self-similar sets

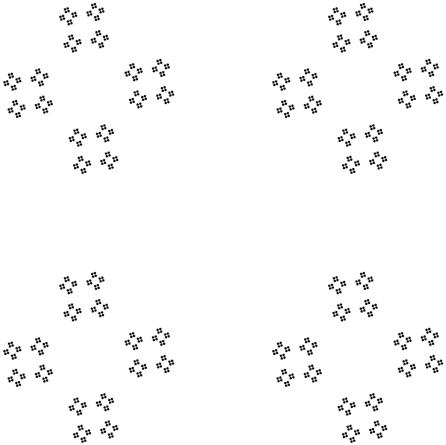


Figure: Another homogeneous self-similar set with rotation

Some planar self-similar sets

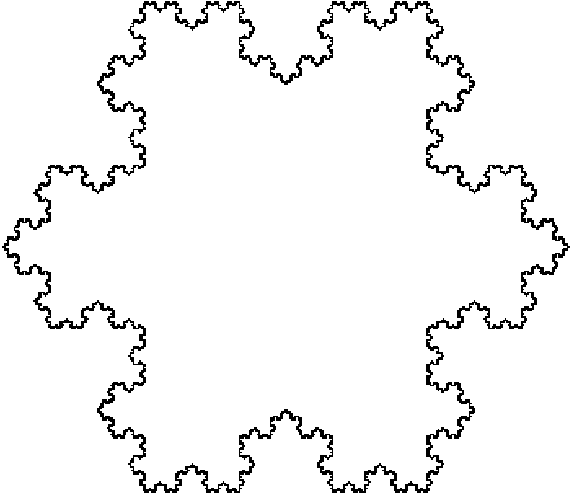


Figure: The von Koch snowflake (not homogeneous)

Box-counting dimension

Definition

- Let $E \subset \mathbb{R}^d$ be a bounded set. Given a small $\delta > 0$, let

$$N_\delta(E)$$

be the smallest number of δ-balls needed to cover E.

- The (upper and lower) box-counting (Minkowski) dimensions of E are

$$\overline{\dim}_B(E) = \limsup_{\delta \to 0} \frac{\log N_\delta(E)}{\log(1/\delta)},$$

$$\underline{\dim}_B(E) = \liminf_{\delta \to 0} \frac{\log N_\delta(E)}{\log(1/\delta)}$$

- If $N_\delta(E) \approx \delta^{-s}$ then $\dim_B(E) = s$.

Box-counting dimension

Definition

- Let $E \subset \mathbb{R}^d$ be a bounded set. Given a small $\delta > 0$, let

$$N_\delta(E)$$

be the smallest number of δ-balls needed to cover E.

- The (upper and lower) box-counting (Minkowski) dimensions of E are

$$\overline{\dim}_B(E) = \limsup_{\delta \to 0} \frac{\log N_\delta(E)}{\log(1/\delta)},$$

$$\underline{\dim}_B(E) = \liminf_{\delta \to 0} \frac{\log N_\delta(E)}{\log(1/\delta)}$$

- If $N_\delta(E) \approx \delta^{-s}$ then $\dim_B(E) = s$.

Box-counting dimension

Definition

- Let $E \subset \mathbb{R}^d$ be a bounded set. Given a small $\delta > 0$, let

$$N_\delta(E)$$

  be the smallest number of δ-balls needed to cover E.
- The (upper and lower) box-counting (Minkowski) dimensions of E are

$$\overline{\dim}_B(E) = \limsup_{\delta \to 0} \frac{\log N_\delta(E)}{\log(1/\delta)},$$

$$\underline{\dim}_B(E) = \liminf_{\delta \to 0} \frac{\log N_\delta(E)}{\log(1/\delta)}$$

- If $N_\delta(E) \approx \delta^{-s}$ then $\dim_B(E) = s$.

Box-counting dimension

Definition

- Let $E \subset \mathbb{R}^d$ be a bounded set. Given a small $\delta > 0$, let

$$N_\delta(E)$$

be the smallest number of δ-balls needed to cover E.

- The (upper and lower) box-counting (Minkowski) dimensions of E are

$$\overline{\dim}_B(E) = \limsup_{\delta\to 0} \frac{\log N_\delta(E)}{\log(1/\delta)},$$
$$\underline{\dim}_B(E) = \liminf_{\delta\to 0} \frac{\log N_\delta(E)}{\log(1/\delta)}$$

- If $N_\delta(E) \approx \delta^{-s}$ then $\dim_B(E) = s$.

Hausdorff dimension

The Hausdorff dimension $\dim_H(A)$ of an arbitrary set $A \subset \mathbb{R}^d$ is a non-negative number that measures the size of A in a reasonable way:

1. $0 \leq \dim_H(A) \leq d$.
2. If A is countable, then $\dim_H(A) = 0$. If A has positive Lebesgue measure, then $\dim_H(A) = d$ (but the reciprocals are not true).
3. If A is a differentiable (or Lipschitz) variety of dimension k, then $\dim_H(A) = k$.
4. If $A \subset B$, then $\dim_H(A) \leq \dim_H(B)$.
5. $\dim_H(\cup_i A_i) = \sup_i \dim(A_i)$.
6. If $f : \mathbb{R}^d \to \mathbb{R}^d$ is (locally) bi-Lipschitz, then $\dim_H(f(A)) = \dim(A)$.
7. $\dim_H(A) \leq \underline{\dim}_B(A) \leq \overline{\dim}_B(A)$.

Hausdorff dimension

The Hausdorff dimension $\dim_H(A)$ of an arbitrary set $A \subset \mathbb{R}^d$ is a non-negative number that measures the size of A in a reasonable way:

1. $0 \leq \dim_H(A) \leq d$.
2. If A is countable, then $\dim_H(A) = 0$. If A has positive Lebesgue measure, then $\dim_H(A) = d$ (but the reciprocals are not true).
3. If A is a differentiable (or Lipschitz) variety of dimension k, then $\dim_H(A) = k$.
4. If $A \subset B$, then $\dim_H(A) \leq \dim_H(B)$.
5. $\dim_H(\cup_i A_i) = \sup_i \dim(A_i)$.
6. If $f : \mathbb{R}^d \to \mathbb{R}^d$ is (locally) bi-Lipschitz, then $\dim_H(f(A)) = \dim(A)$.
7. $\dim_H(A) \leq \underline{\dim}_B(A) \leq \overline{\dim}_B(A)$.

Hausdorff dimension

The Hausdorff dimension $\dim_H(A)$ of an arbitrary set $A \subset \mathbb{R}^d$ is a non-negative number that measures the size of A in a reasonable way:

1. $0 \le \dim_H(A) \le d$.
2. If A is countable, then $\dim_H(A) = 0$. If A has positive Lebesgue measure, then $\dim_H(A) = d$ (but the reciprocals are not true).
3. If A is a differentiable (or Lipschitz) variety of dimension k, then $\dim_H(A) = k$.
4. If $A \subset B$, then $\dim_H(A) \le \dim_H(B)$.
5. $\dim_H(\cup_i A_i) = \sup_i \dim(A_i)$.
6. If $f : \mathbb{R}^d \to \mathbb{R}^d$ is (locally) bi-Lipschitz, then $\dim_H(f(A)) = \dim(A)$.
7. $\dim_H(A) \le \underline{\dim}_B(A) \le \overline{\dim}_B(A)$.

Hausdorff dimension

The Hausdorff dimension $\dim_H(A)$ of an arbitrary set $A \subset \mathbb{R}^d$ is a non-negative number that measures the size of A in a reasonable way:

1. $0 \leq \dim_H(A) \leq d$.
2. If A is countable, then $\dim_H(A) = 0$. If A has positive Lebesgue measure, then $\dim_H(A) = d$ (but the reciprocals are not true).
3. If A is a differentiable (or Lipschitz) variety of dimension k, then $\dim_H(A) = k$.
4. If $A \subset B$, then $\dim_H(A) \leq \dim_H(B)$.
5. $\dim_H(\cup_i A_i) = \sup_i \dim(A_i)$.
6. If $f : \mathbb{R}^d \to \mathbb{R}^d$ is (locally) bi-Lipschitz, then $\dim_H(f(A)) = \dim(A)$.
7. $\dim_H(A) \leq \underline{\dim}_B(A) \leq \overline{\dim}_B(A)$.

Hausdorff dimension

The Hausdorff dimension $\dim_H(A)$ of an arbitrary set $A \subset \mathbb{R}^d$ is a non-negative number that measures the size of A in a reasonable way:

1. $0 \leq \dim_H(A) \leq d$.
2. If A is countable, then $\dim_H(A) = 0$. If A has positive Lebesgue measure, then $\dim_H(A) = d$ (but the reciprocals are not true).
3. If A is a differentiable (or Lipschitz) variety of dimension k, then $\dim_H(A) = k$.
4. If $A \subset B$, then $\dim_H(A) \leq \dim_H(B)$.
5. $\dim_H(\cup_i A_i) = \sup_i \dim(A_i)$.
6. If $f : \mathbb{R}^d \to \mathbb{R}^d$ is (locally) bi-Lipschitz, then $\dim_H(f(A)) = \dim(A)$.
7. $\dim_H(A) \leq \underline{\dim}_B(A) \leq \overline{\dim}_B(A)$.

Hausdorff dimension

The Hausdorff dimension $\dim_H(A)$ of an arbitrary set $A \subset \mathbb{R}^d$ is a non-negative number that measures the size of A in a reasonable way:

1. $0 \leq \dim_H(A) \leq d$.
2. If A is countable, then $\dim_H(A) = 0$. If A has positive Lebesgue measure, then $\dim_H(A) = d$ (but the reciprocals are not true).
3. If A is a differentiable (or Lipschitz) variety of dimension k, then $\dim_H(A) = k$.
4. If $A \subset B$, then $\dim_H(A) \leq \dim_H(B)$.
5. $\dim_H(\cup_i A_i) = \sup_i \dim(A_i)$.
6. If $f : \mathbb{R}^d \to \mathbb{R}^d$ is (locally) bi-Lipschitz, then $\dim_H(f(A)) = \dim(A)$.
7. $\dim_H(A) \leq \underline{\dim}_B(A) \leq \overline{\dim}_B(A)$.

Hausdorff dimension

The Hausdorff dimension $\dim_H(A)$ of an arbitrary set $A \subset \mathbb{R}^d$ is a non-negative number that measures the size of A in a reasonable way:

1. $0 \leq \dim_H(A) \leq d$.
2. If A is countable, then $\dim_H(A) = 0$. If A has positive Lebesgue measure, then $\dim_H(A) = d$ (but the reciprocals are not true).
3. If A is a differentiable (or Lipschitz) variety of dimension k, then $\dim_H(A) = k$.
4. If $A \subset B$, then $\dim_H(A) \leq \dim_H(B)$.
5. $\dim_H(\cup_i A_i) = \sup_i \dim(A_i)$.
6. If $f : \mathbb{R}^d \to \mathbb{R}^d$ is (locally) bi-Lipschitz, then $\dim_H(f(A)) = \dim(A)$.
7. $\dim_H(A) \leq \underline{\dim}_B(A) \leq \overline{\dim}_B(A)$.

Hausdorff dimension: definition

- Given $A \subset \mathbb{R}^d$, let

$$\mathcal{H}^s(A) = \inf \left\{ \sum_i r_i^s : A \subset \bigcup_i B(x_i, r_i) \right\}$$

- The function $s \mapsto \mathcal{H}^s(A)$ is decreasing, and is 0 if $s > d$ (it is 0 for $s = d$ exactly when A has zero Lebesgue measure).
- $$\dim_{\mathrm{H}}(A) = \inf\{s : \mathcal{H}^s(A) = 0\}.$$

Hausdorff dimension: definition

- Given $A \subset \mathbb{R}^d$, let

$$\mathcal{H}^s(A) = \inf \left\{ \sum_i r_i^s : A \subset \bigcup_i B(x_i, r_i) \right\}$$

- The function $s \mapsto \mathcal{H}^s(A)$ is decreasing, and is 0 if $s > d$ (it is 0 for $s = d$ exactly when A has zero Lebesgue measure).
- $$\dim_{\mathrm{H}}(A) = \inf\{s : \mathcal{H}^s(A) = 0\}.$$

Hausdorff dimension: definition

- Given $A \subset \mathbb{R}^d$, let

$$\mathcal{H}^s(A) = \inf \left\{ \sum_i r_i^s : A \subset \bigcup_i B(x_i, r_i) \right\}$$

- The function $s \mapsto \mathcal{H}^s(A)$ is decreasing, and is 0 if $s > d$ (it is 0 for $s = d$ exactly when A has zero Lebesgue measure).
- $$\dim_{\mathrm{H}}(A) = \inf\{s : \mathcal{H}^s(A) = 0\}.$$

Dimensions of self-similar sets

- Let $E = \cup_{i=1}^{m} f_i(E)$, where the similarities f_i have the same contraction ratio r.
- It always holds that $\dim_H(E) = \underline{\dim}_B(E) = \overline{\dim}_B(E)$.
- If the pieces $f_i(E)$ "do not overlap too much" (open set condition, etc), then

$$\dim_H(E) = \frac{\log m}{\log(1/r)}.$$

Dimensions of self-similar sets

- Let $E = \cup_{i=1}^{m} f_i(E)$, where the similarities f_i have the same contraction ratio r.
- It always holds that $\dim_{\mathrm{H}}(E) = \underline{\dim}_{\mathrm{B}}(E) = \overline{\dim}_{\mathrm{B}}(E)$.
- If the pieces $f_i(E)$ "do not overlap too much" (open set condition, etc), then

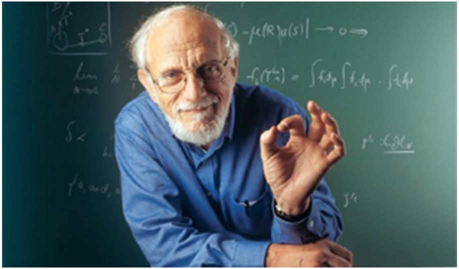
$$\dim_{\mathrm{H}}(E) = \frac{\log m}{\log(1/r)}.$$

Dimensions of self-similar sets

- Let $E = \cup_{i=1}^{m} f_i(E)$, where the similarities f_i have the same contraction ratio r.
- It always holds that $\dim_H(E) = \underline{\dim}_B(E) = \overline{\dim}_B(E)$.
- If the pieces $f_i(E)$ "do not overlap too much" (open set condition, etc), then

$$\dim_H(E) = \frac{\log m}{\log(1/r)}.$$

Furstenberg's conjectures



In the 1960s, Furstenberg stated a number of conjectures on the Hausdorff dimensions of various fractals sets that give insight into dynamics/arithmetic (particularly about expansions to an integer base).

The one-dimensional Sierpiński gasket G

Furstenberg's conjecture on G

$$P_\theta(x) = \langle x, \theta \rangle \quad (\theta \in S^1).$$

Conjecture (H. Furstenberg 1960s?)

For every θ with irrational slope, $\dim_H(P_\theta G) = 1$.

Theorem (M. Hochman + B. Solomyak 2012)

Furstenberg's conjecture is true.

Furstenberg's conjecture on G

$$P_\theta(x) = \langle x, \theta \rangle \quad (\theta \in S^1).$$

Conjecture (H. Furstenberg 1960s?)

For every θ with irrational slope, $\dim_H(P_\theta G) = 1$*.*

Theorem (M. Hochman + B. Solomyak 2012)

Furstenberg's conjecture is true.

Fursteberg's slicing conjecture

Conjecture (H. Furstenberg 1969)

Let $A, B \subset [0,1] \subset \mathbb{R}$ *be closed and invariant under* T_p, T_q *respectively, where* $p \not\sim q$ *(meaning* $\log p / \log q \notin \mathbb{Q}$*). Then*

$$\dim_H(A \cap g(B)) \leq \max(\dim_H(A) + \dim_H(B) - 1, 0)$$

for all non-constant affine maps g*.*

Remark

This conjecture express in geometric terms the heuristic principle that "expansions to bases p and q have no common structure".

Theorem (P.S./ M. Wu 2019)

Furstenberg's slicing conjecture holds.

Fursteberg's slicing conjecture

Conjecture (H. Furstenberg 1969)

Let $A, B \subset [0,1] \subset \mathbb{R}$ be closed and invariant under T_p, T_q respectively, where $p \not\sim q$ (meaning $\log p / \log q \notin \mathbb{Q}$). Then

$$\dim_{\mathrm{H}}(A \cap g(B)) \leq \max(\dim_{\mathrm{H}}(A) + \dim_{\mathrm{H}}(B) - 1, 0)$$

for all non-constant affine maps g.

Remark

This conjecture express in geometric terms the heuristic principle that "expansions to bases p and q have no common structure".

Theorem (P.S./ M. Wu 2019)

Furstenberg's slicing conjecture holds.

Fursteberg's slicing conjecture

Conjecture (H. Furstenberg 1969)

Let $A, B \subset [0,1] \subset \mathbb{R}$ be closed and invariant under T_p, T_q respectively, where $p \not\sim q$ (meaning $\log p / \log q \notin \mathbb{Q}$). Then

$$\dim_H(A \cap g(B)) \leq \max(\dim_H(A) + \dim_H(B) - 1, 0)$$

for all non-constant affine maps g.

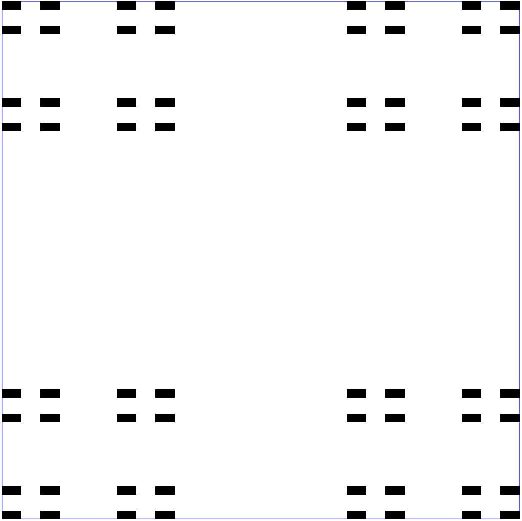
Remark

This conjecture express in geometric terms the heuristic principle that "expansions to bases p and q have no common structure".

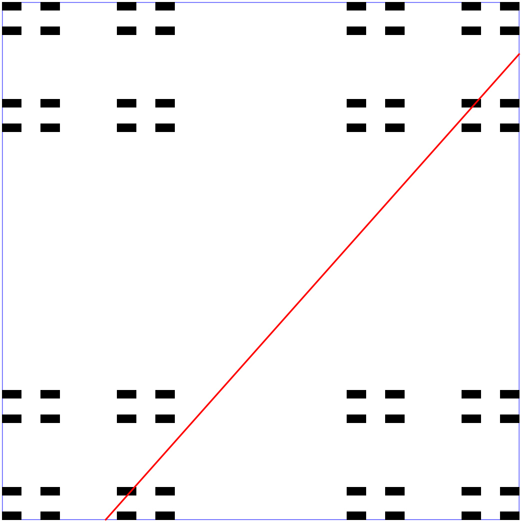
Theorem (P.S./ M. Wu 2019)

Furstenberg's slicing conjecture holds.

Furstenberg's slicing conjecture in pictures



Furstenberg's slicing conjecture in pictures



Linear slices of self-affine sets

Theorem (P.S. / Meng Wu 2019)

Let A, B be closed and p, q-Cantor sets with $p \not\sim q$. Then

$$\dim_H(A \times B \cap \ell) \le \max(\dim_H(A) + \dim_H(B) - 1, 0)$$

for all non vertical/horizontal lines.

- The two methods are completely different. Meng Wu uses ergodic theory and CP-chains. My method relies on additive combinatorics.
- The set $A \times B$ is self-affine; it is made up of affine images of itself.
- $A \times B$ is invariant under $T_{p,q}(x, y) = (px \bmod 1, qx \bmod 1)$ on the torus. Very recently, A. Algom and M. Wu extended this result to general closed $T_{p,q}$-invariant sets.
- The theorem also holds for real analytic curves (other than horizontal or vertical lines).

Linear slices of self-affine sets

Theorem (P.S. / Meng Wu 2019)

Let A, B be closed and p, q-Cantor sets with $p \not\sim q$. Then

$$\dim_H(A \times B \cap \ell) \leq \max(\dim_H(A) + \dim_H(B) - 1, 0)$$

for all non vertical/horizontal lines.

- The two methods are completely different. Meng Wu uses ergodic theory and CP-chains. My method relies on additive combinatorics.
- The set $A \times B$ is self-affine; it is made up of affine images of itself.
- $A \times B$ is invariant under $T_{p,q}(x, y) = (px \bmod 1, qx \bmod 1)$ on the torus. Very recently, A. Algom and M. Wu extended this result to general closed $T_{p,q}$-invariant sets.
- The theorem also holds for real analytic curves (other than horizontal or vertical lines).

Linear slices of self-affine sets

Theorem (P.S. / Meng Wu 2019)

Let A, B be closed and p, q-Cantor sets with $p \not\sim q$. Then

$$\dim_H(A \times B \cap \ell) \leq \max(\dim_H(A) + \dim_H(B) - 1, 0)$$

for all non vertical/horizontal lines.

- The two methods are completely different. Meng Wu uses ergodic theory and CP-chains. My method relies on additive combinatorics.
- The set $A \times B$ is self-affine; it is made up of affine images of itself.
- $A \times B$ is invariant under $T_{p,q}(x, y) = (px \bmod 1, qx \bmod 1)$ on the torus. Very recently, A. Algom and M. Wu extended this result to general closed $T_{p,q}$-invariant sets.
- The theorem also holds for real analytic curves (other than horizontal or vertical lines).

Linear slices of self-affine sets

Theorem (P.S. / Meng Wu 2019)

Let A, B be closed and p, q-Cantor sets with $p \not\sim q$. Then

$$\dim_H(A \times B \cap \ell) \leq \max(\dim_H(A) + \dim_H(B) - 1, 0)$$

for all non vertical/horizontal lines.

- The two methods are completely different. Meng Wu uses ergodic theory and CP-chains. My method relies on additive combinatorics.
- The set $A \times B$ is self-affine; it is made up of affine images of itself.
- $A \times B$ is invariant under $T_{p,q}(x, y) = (px \bmod 1, qx \bmod 1)$ on the torus. Very recently, A. Algom and M. Wu extended this result to general closed $T_{p,q}$-invariant sets.
- The theorem also holds for real analytic curves (other than horizontal or vertical lines).

Linear slices of self-affine sets

Theorem (P.S. / Meng Wu 2019)

Let A, B be closed and p, q-Cantor sets with $p \not\sim q$. Then

$$\dim_H(A \times B \cap \ell) \leq \max(\dim_H(A) + \dim_H(B) - 1, 0)$$

for all non vertical/horizontal lines.

- The two methods are completely different. Meng Wu uses ergodic theory and CP-chains. My method relies on additive combinatorics.
- The set $A \times B$ is self-affine; it is made up of affine images of itself.
- $A \times B$ is invariant under $T_{p,q}(x, y) = (px \bmod 1, qx \bmod 1)$ on the torus. Very recently, A. Algom and M. Wu extended this result to general closed $T_{p,q}$-invariant sets.
- The theorem also holds for real analytic curves (other than horizontal or vertical lines).

Interpolating between the two conjectures

- There are two main differences between the two conjectures:
  1. One refers to projections, the other to slices.
  2. One is about self-similar sets (one basis, T_3), the other about self-affine sets (two bases, $T_{p,q}$).
- We can interpolate by asking about projections of $T_{p,q}$-invariant sets or about slices of T_p-invariant sets.

Interpolating between the two conjectures

- There are two main differences between the two conjectures:
  1. One refers to projections, the other to slices.
  2. One is about self-similar sets (one basis, T_3), the other about self-affine sets (two bases, $T_{p,q}$).
- We can interpolate by asking about projections of $T_{p,q}$-invariant sets or about slices of T_p-invariant sets.

Interpolating between the two conjectures

- There are two main differences between the two conjectures:
  1. One refers to projections, the other to slices.
  2. One is about self-similar sets (one basis, T_3), the other about self-affine sets (two bases, $T_{p,q}$).
- We can interpolate by asking about projections of $T_{p,q}$-invariant sets or about slices of T_p-invariant sets.

Interpolating between the two conjectures

- There are two main differences between the two conjectures:
  1. One refers to projections, the other to slices.
  2. One is about self-similar sets (one basis, T_3), the other about self-affine sets (two bases, $T_{p,q}$).
- We can interpolate by asking about projections of $T_{p,q}$-invariant sets or about slices of T_p-invariant sets.

Furstenberg's sumset conjecture

Conjecture (H. Furstenberg 1960s)

If A, B are closed and T_p, T_q-invariant then

$$\dim_{\mathrm{H}}(P_\theta(A \times B)) = \min(\dim_{\mathrm{H}}(A) + \dim_{\mathrm{H}}(B), 1).$$

for all $\theta \notin \{0, \pi/2\}$.

Theorem (M. Hochman and P.S. 2012)

The conjecture holds.

Remark

It can be shown that the slicing conjecture is formally stronger than the sumset conjecture. In particular, the two proofs to the slicing conjecture give two new proofs for the projection conjecture.

Furstenberg's sumset conjecture

Conjecture (H. Furstenberg 1960s)

If A, B are closed and T_p, T_q-invariant then

$$\dim_H(P_\theta(A \times B)) = \min(\dim_H(A) + \dim_H(B), 1).$$

for all $\theta \notin \{0, \pi/2\}$.

Theorem (M. Hochman and P.S. 2012)

The conjecture holds.

Remark

It can be shown that the slicing conjecture is formally stronger than the sumset conjecture. In particular, the two proofs to the slicing conjecture give two new proofs for the projection conjecture.

Furstenberg's sumset conjecture

Conjecture (H. Furstenberg 1960s)

If A, B are closed and T_p, T_q-invariant then

$$\dim_H(P_\theta(A \times B)) = \min(\dim_H(A) + \dim_H(B), 1).$$

for all $\theta \notin \{0, \pi/2\}$.

Theorem (M. Hochman and P.S. 2012)

The conjecture holds.

Remark

It can be shown that the slicing conjecture is formally stronger than the sumset conjecture. In particular, the two proofs to the slicing conjecture give two new proofs for the projection conjecture.

Slices of T_n-invariant sets

Theorem (P.S. 2019)

Let $E \subset [0,1]^2$ be closed and T_p-invariant (for example, the one dim. Sierpiński gasket).

Then for every line ℓ with irrational slope,

$$\dim_H(E \cap \ell) \leq \overline{\dim}_B(E \cap \ell) \leq \max(\dim_H(E) - 1, 0).$$

In fact, if θ has irrational slope, then for every $s > \max(\dim_H(E) - 1, 0)$, the intersection $E \cap \ell$ can be covered by $C_{\theta,s} r^{-s}$ balls of radius r for all lines ℓ in direction θ.

Note that $C_{\theta,s}$ does not depend on the line, only on the angle.

Slices of T_n-invariant sets

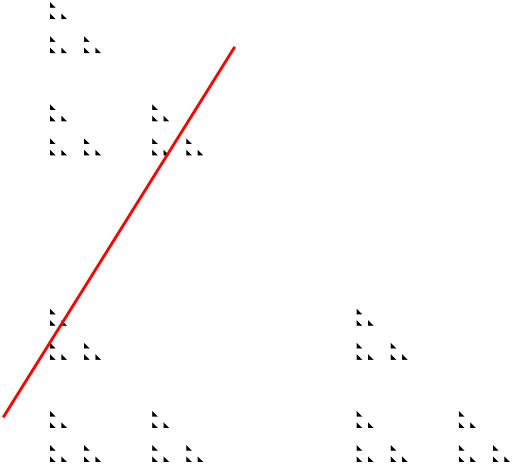


Figure: Each line with irrational slope intersects a sub-exponential number of small triangles

Slices of T_n-invariant sets

Remarks

- *For (infinitely many) rational directions this is not true: in a direction for which two pieces in the construction have an exact overlap, the slice has larger dimension.*
- *Meng Wu's approach does not work in this setting. The proof uses additive combinatorics and multifractal analysis, no ergodic theory.*

Corollary

Let G be the one-dim Sierpiński gasket (or any T_p-invariant set of dimension ≤ 1). Then for all irrational θ,

$$\dim_H(P_\theta F) = \dim_H(F) \quad \textit{for all } F \subset G.$$

Slices of T_n-invariant sets

Remarks

- *For (infinitely many) rational directions this is not true: in a direction for which two pieces in the construction have an exact overlap, the slice has larger dimension.*
- *Meng Wu's approach does not work in this setting. The proof uses additive combinatorics and multifractal analysis, no ergodic theory.*

Corollary

Let G be the one-dim Sierpiński gasket (or any T_p-invariant set of dimension ≤ 1). Then for all irrational θ,

$$\dim_H(P_\theta F) = \dim_H(F) \quad \text{for all } F \subset G.$$

Slices of T_n-invariant sets

Remarks

- *For (infinitely many) rational directions this is not true: in a direction for which two pieces in the construction have an exact overlap, the slice has larger dimension.*
- *Meng Wu's approach does not work in this setting. The proof uses additive combinatorics and multifractal analysis, no ergodic theory.*

Corollary

Let G be the one-dim Sierpiński gasket (or any T_p-invariant set of dimension ≤ 1). Then for all irrational θ,

$$\dim_H(P_\theta F) = \dim_H(F) \quad \text{for all } F \subset G.$$

Slices of T_n-invariant sets

Remarks

- *For (infinitely many) rational directions this is not true: in a direction for which two pieces in the construction have an exact overlap, the slice has larger dimension.*
- *Meng Wu's approach does not work in this setting. The proof uses additive combinatorics and multifractal analysis, no ergodic theory.*

Corollary

Let G be the one-dim Sierpiński gasket (or any T_p-invariant set of dimension ≤ 1). Then for all irrational θ,

$$\dim_H(P_\theta F) = \dim_H(F) \quad \textit{for all } F \subset G.$$

Slices of homogeneous self-similar sets

Theorem

Let $E \subset \mathbb{R}^2$ be a homogeneous self-similar set with OSC.

1. *(P.S./M. Wu 2019) Suppose the rotation is irrational. Then*

$$\dim_H(E \cap \ell) \leq \overline{\dim}_B(E \cap \ell) \leq \max(\dim_H(E) - 1, 0)$$

   for every line ℓ.

2. *(P.S. 2019) If the rotation is rational, there exists a set Θ of directions of zero Hausdorff (and packing) dimension such that*

$$\dim_H(E \cap \ell) \leq \overline{\dim}_B(E \cap \ell) \leq \max(\dim_H(E) - 1, 0)$$

   for all lines ℓ with direction not in Θ.

Slices of homogeneous self-similar sets

Theorem

Let $E \subset \mathbb{R}^2$ be a homogeneous self-similar set with OSC.

1. *(P.S./M. Wu 2019) Suppose the rotation is irrational. Then*

$$\dim_H(E \cap \ell) \leq \overline{\dim}_B(E \cap \ell) \leq \max(\dim_H(E) - 1, 0)$$

for every line ℓ.

2. *(P.S. 2019) If the rotation is rational, there exists a set Θ of directions of zero Hausdorff (and packing) dimension such that*

$$\dim_H(E \cap \ell) \leq \overline{\dim}_B(E \cap \ell) \leq \max(\dim_H(E) - 1, 0)$$

for all lines ℓ with direction not in Θ.

Slices of homogeneous self-similar sets

Theorem

Let $E \subset \mathbb{R}^2$ be a homogeneous self-similar set with OSC.

1. *(P.S./M. Wu 2019) Suppose the rotation is irrational. Then*

$$\dim_H(E \cap \ell) \le \overline{\dim}_B(E \cap \ell) \le \max(\dim_H(E) - 1, 0)$$

   for every line ℓ.

2. *(P.S. 2019) If the rotation is rational, there exists a set Θ of directions of zero Hausdorff (and packing) dimension such that*

$$\dim_H(E \cap \ell) \le \overline{\dim}_B(E \cap \ell) \le \max(\dim_H(E) - 1, 0)$$

   for all lines ℓ with direction not in Θ.

Intersections with curves

Corollary (P.S. 2020?)

Let $E \subset \mathbb{R}^2$ be a homogeneous self-similar set with OSC and let σ be a C^1 curve.

1. *If E has irrational rotation, then*

$$\dim_H(E \cap \sigma) \leq \overline{\dim}_B(E \cap \sigma) \leq \max(\dim_H(E) - 1, 0).$$

2. *If E has rational rotation, then the same holds provided the set of times t such that $\sigma'(t)$ has rational slope has zero Hausdorff dimension. In particular, it holds for any non-linear real-analytic curve.*
3. *If the curve is only differentiable, the same still holds for Hausdorff dimension (and even packing dimension).*
4. *On the other hand, this is wildly false for Lipschitz curves (any set of box dimension < 1 can be covered by a Lipschitz curve).*

Intersections with curves

Corollary (P.S. 2020?)

Let $E \subset \mathbb{R}^2$ be a homogeneous self-similar set with OSC and let σ be a C^1 curve.

1. *If E has irrational rotation, then*

$$\dim_H(E \cap \sigma) \le \overline{\dim}_B(E \cap \sigma) \le \max(\dim_H(E) - 1, 0).$$

2. *If E has rational rotation, then the same holds provided the set of times t such that $\sigma'(t)$ has rational slope has zero Hausdorff dimension. In particular, it holds for any non-linear real-analytic curve.*
3. *If the curve is only differentiable, the same still holds for Hausdorff dimension (and even packing dimension).*
4. *On the other hand, this is wildly false for Lipschitz curves (any set of box dimension < 1 can be covered by a Lipschitz curve).*

Intersections with curves

Corollary (P.S. 2020?)

Let $E \subset \mathbb{R}^2$ be a homogeneous self-similar set with OSC and let σ be a C^1 curve.

1. *If E has irrational rotation, then*

$$\dim_H(E \cap \sigma) \leq \overline{\dim}_B(E \cap \sigma) \leq \max(\dim_H(E) - 1, 0).$$

2. *If E has rational rotation, then the same holds provided the set of times t such that $\sigma'(t)$ has rational slope has zero Hausdorff dimension. In particular, it holds for any non-linear real-analytic curve.*
3. *If the curve is only differentiable, the same still holds for Hausdorff dimension (and even packing dimension).*
4. *On the other hand, this is wildly false for Lipschitz curves (any set of box dimension < 1 can be covered by a Lipschitz curve).*

Intersections with curves

Corollary (P.S. 2020?)

Let $E \subset \mathbb{R}^2$ be a homogeneous self-similar set with OSC and let σ be a C^1 curve.

1. *If E has irrational rotation, then*

$$\dim_H(E \cap \sigma) \le \overline{\dim}_B(E \cap \sigma) \le \max(\dim_H(E) - 1, 0).$$

2. *If E has rational rotation, then the same holds provided the set of times t such that $\sigma'(t)$ has rational slope has zero Hausdorff dimension. In particular, it holds for any non-linear real-analytic curve.*
3. *If the curve is only differentiable, the same still holds for Hausdorff dimension (and even packing dimension).*
4. *On the other hand, this is wildly false for Lipschitz curves (any set of box dimension < 1 can be covered by a Lipschitz curve).*

Intersections with curves

Corollary (P.S. 2020?)

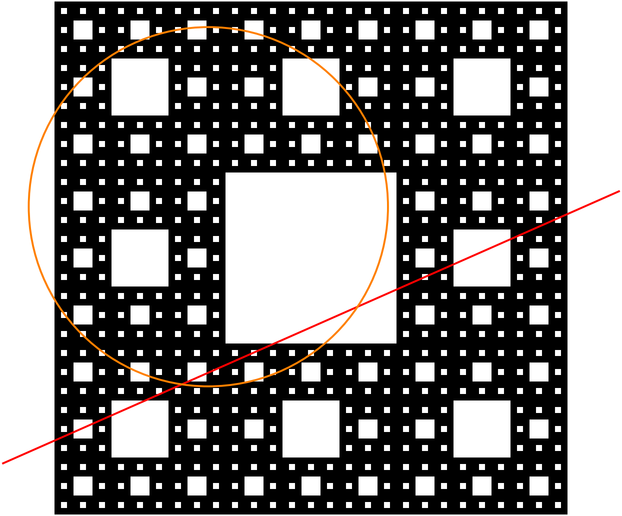
Let $E \subset \mathbb{R}^2$ be a homogeneous self-similar set with OSC and let σ be a C^1 curve.

1. *If E has irrational rotation, then*

$$\dim_H(E \cap \sigma) \leq \overline{\dim}_B(E \cap \sigma) \leq \max(\dim_H(E) - 1, 0).$$

2. *If E has rational rotation, then the same holds provided the set of times t such that $\sigma'(t)$ has rational slope has zero Hausdorff dimension. In particular, it holds for any non-linear real-analytic curve.*
3. *If the curve is only differentiable, the same still holds for Hausdorff dimension (and even packing dimension).*
4. *On the other hand, this is wildly false for Lipschitz curves (any set of box dimension < 1 can be covered by a Lipschitz curve).*

Slices of the Sierpiński carpet



Tube-null sets

Definition

- *A tube (in the plane) is an ε-neighborhood of a line. The width $w(T)$ of the tube T is ε.*
- *A set $E \subset \mathbb{R}^2$ is tube-null if, for any $\varepsilon > 0$, it can be covered by a countable union of tubes $\{T_i\}$ with $\sum_i w(T_i) < \varepsilon$.*

Tube-null sets

Definition

- *A* tube *(in the plane) is an* ε*-neighborhood of a line. The* width $w(T)$ *of the tube* T *is* ε*.*
- *A set* $E \subset \mathbb{R}^2$ *is* tube-null *if, for any* $\varepsilon > 0$*, it can be covered by a countable union of tubes* $\{T_i\}$ *with* $\sum_i w(T_i) < \varepsilon$*.*

Tube-null sets

Definition

- *A tube (in the plane) is an ε-neighborhood of a line. The width $w(T)$ of the tube T is ε.*
- *A set $E \subset \mathbb{R}^2$ is tube-null if, for any $\varepsilon > 0$, it can be covered by a countable union of tubes $\{T_i\}$ with $\sum_i w(T_i) < \varepsilon$.*

Properties of tube-null sets

- Any tube-null set is Lebesgue-null. (The converse does not hold.)
- A subset of a tube-null set is tube-null.
- A countable union of tube-null sets is tube-null.
- If $P_\theta E$ is Lebesgue null (in $\mathbb{R}$) for some θ, then E is tube-null.
- There are tube-null sets of Hausdorff dimension 2: take $A \times \mathbb{R}$, where A has zero Lebesgue measure and Hausdorff dimension 1.
- (Carbery-Soria-Vargas) Sets of σ-finite 1-dim. Hausdorff measure are tube-null (idea: decompose them as a union of a purely unrectifiable and a rectifiable set, and use Besicovitch's projection theorem for the unrectifiable part).

Properties of tube-null sets

- Any tube-null set is Lebesgue-null. (The converse does not hold.)
- A subset of a tube-null set is tube-null.
- A countable union of tube-null sets is tube-null.
- If $P_\theta E$ is Lebesgue null (in $\mathbb{R}$) for some θ, then E is tube-null.
- There are tube-null sets of Hausdorff dimension 2: take $A \times \mathbb{R}$, where A has zero Lebesgue measure and Hausdorff dimension 1.
- (Carbery-Soria-Vargas) Sets of σ-finite 1-dim. Hausdorff measure are tube-null (idea: decompose them as a union of a purely unrectifiable and a rectifiable set, and use Besicovitch's projection theorem for the unrectifiable part).

Properties of tube-null sets

- Any tube-null set is Lebesgue-null. (The converse does not hold.)
- A subset of a tube-null set is tube-null.
- A countable union of tube-null sets is tube-null.
- If $P_\theta E$ is Lebesgue null (in $\mathbb{R}$) for some θ, then E is tube-null.
- There are tube-null sets of Hausdorff dimension 2: take $A \times \mathbb{R}$, where A has zero Lebesgue measure and Hausdorff dimension 1.
- (Carbery-Soria-Vargas) Sets of σ-finite 1-dim. Hausdorff measure are tube-null (idea: decompose them as a union of a purely unrectifiable and a rectifiable set, and use Besicovitch's projection theorem for the unrectifiable part).

Properties of tube-null sets

- Any tube-null set is Lebesgue-null. (The converse does not hold.)
- A subset of a tube-null set is tube-null.
- A countable union of tube-null sets is tube-null.
- If $P_\theta E$ is Lebesgue null (in $\mathbb{R}$) for some θ, then E is tube-null.
- There are tube-null sets of Hausdorff dimension 2: take $A \times \mathbb{R}$, where A has zero Lebesgue measure and Hausdorff dimension 1.
- (Carbery-Soria-Vargas) Sets of σ-finite 1-dim. Hausdorff measure are tube-null (idea: decompose them as a union of a purely unrectifiable and a rectifiable set, and use Besicovitch's projection theorem for the unrectifiable part).

Properties of tube-null sets

- Any tube-null set is Lebesgue-null. (The converse does not hold.)
- A subset of a tube-null set is tube-null.
- A countable union of tube-null sets is tube-null.
- If $P_\theta E$ is Lebesgue null (in $\mathbb{R}$) for some θ, then E is tube-null.
- There are tube-null sets of Hausdorff dimension 2: take $A \times \mathbb{R}$, where A has zero Lebesgue measure and Hausdorff dimension 1.
- (Carbery-Soria-Vargas) Sets of σ-finite 1-dim. Hausdorff measure are tube-null (idea: decompose them as a union of a purely unrectifiable and a rectifiable set, and use Besicovitch's projection theorem for the unrectifiable part).

Properties of tube-null sets

- Any tube-null set is Lebesgue-null. (The converse does not hold.)
- A subset of a tube-null set is tube-null.
- A countable union of tube-null sets is tube-null.
- If $P_\theta E$ is Lebesgue null (in $\mathbb{R}$) for some θ, then E is tube-null.
- There are tube-null sets of Hausdorff dimension 2: take $A \times \mathbb{R}$, where A has zero Lebesgue measure and Hausdorff dimension 1.
- (Carbery-Soria-Vargas) Sets of σ-finite 1-dim. Hausdorff measure are tube-null (idea: decompose them as a union of a purely unrectifiable and a rectifiable set, and use Besicovitch's projection theorem for the unrectifiable part).

Dimension of sets which are **not** tube-null

Question (Carbery)

What is $\inf\{\dim_H(K) : K$ *is* ***not*** *tube null* $\}$*? For what dimensions are there non-tube-null Ahlfors-regular sets?*

Theorem (P. S.-V. Suomala 2011)

There are (random) sets of any dimension ≥ 1 *which are* ***not*** *tube null, and they can be taken to be Ahlfors-regular if the dimension is* > 1.

Dimension of sets which are **not** tube-null

Question (Carbery)

What is $\inf\{\dim_{\mathrm{H}}(K) : K$ *is **not** tube null* $\}$*? For what dimensions are there non-tube-null Ahlfors-regular sets?*

Theorem (P. S.-V. Suomala 2011)

There are (random) sets of any dimension ≥ 1 *which are **not** tube null, and they can be taken to be Ahlfors-regular if the dimension is* > 1*.*

The localization problem

Definition

Given $f \in L^2(\mathbb{R}^d)$, let

$$S_R f(x) = \int_{|\xi|<R} \widehat{f}(\xi) e^{2\pi i x\cdot\xi} d\xi$$

be the localization of f to frequencies of modulus $\leq R$.

Open problem

Is it true that for any $f \in L^2$,

$$f(x) = \lim_{R\to\infty} S_R f(x) \quad \textit{for almost every } x\ ?.$$

Remark

Famous result of Carleson in dimension 1. Open in higher dimensions.

The localization problem

Definition

Given $f \in L^2(\mathbb{R}^d)$*, let*

$$S_R f(x) = \int_{|\xi|<R} \widehat{f}(\xi) e^{2\pi i x\cdot\xi} d\xi$$

be the localization of f *to frequencies of modulus* $\leq R$.

Open problem

Is it true that for any $f \in L^2$,

$$f(x) = \lim_{R\to\infty} S_R f(x) \quad \text{for almost every } x \text{ ?.}$$

Remark

Famous result of Carleson in dimension 1*. Open in higher dimensions.*

The localization problem

Definition

Given $f \in L^2(\mathbb{R}^d)$, let

$$S_R f(x) = \int_{|\xi|<R} \widehat{f}(\xi) e^{2\pi i x\cdot\xi} d\xi$$

be the localization of f to frequencies of modulus $\leq R$.

Open problem

Is it true that for any $f \in L^2$,

$$f(x) = \lim_{R\to\infty} S_R f(x) \quad \textit{for almost every } x\ ?.$$

Remark

Famous result of Carleson in dimension 1. Open in higher dimensions.

Localization and tube-null sets

Theorem (Carbery-Soria 1988)

Let Ω be a compact domain (for example unit disk). If $f \in L^2(\mathbb{R}^2)$ and $\mathrm{supp}(f) \cap \Omega = \emptyset$, then

$$S_R f(x) \to 0 \quad \text{for almost every } x \in \Omega.$$

Theorem (Carbery, Soria and Vargas 2007)

If $E \subset \Omega$ is tube-null, then there is $f \in L^2(\mathbb{R}^2)$ with $\mathrm{supp}(f) \cap \Omega = \emptyset$ such that

$$S_R f(x) \not\to 0 \quad \text{for all } x \in E.$$

Localization and tube-null sets

Theorem (Carbery-Soria 1988)

Let Ω be a compact domain (for example unit disk). If $f \in L^2(\mathbb{R}^2)$ and supp$(f) \cap \Omega = \emptyset$, then

$$S_R f(x) \to 0 \quad \text{for almost every } x \in \Omega.$$

Theorem (Carbery, Soria and Vargas 2007)

If $E \subset \Omega$ is tube-null, then there is $f \in L^2(\mathbb{R}^2)$ with supp$(f) \cap \Omega = \emptyset$ such that

$$S_R f(x) \not\to 0 \quad \text{for all } x \in E.$$

Which sets are tube-null?

- There is no (non-trivial) connection between Hausdorff dimension and tube-nullity: there are tube-null sets of dimension 2 and sets of dimension 1 which are **not** tube-null. Still, intuitively, sets of large dimension should have more difficulty being tube-null.
- If we can decompose E into countably many pieces E_θ such that $P_\theta E_\theta$ is Lebesgue-null, then E is tube-null.
- There were very few non-trivial examples of tube-null sets of large dimension. In particular, it seems reasonable to ask which self-similar sets are tube-null.

Theorem (V. Harangi 2011)

The von Koch snowflake is tube-null.

Which sets are tube-null?

- There is no (non-trivial) connection between Hausdorff dimension and tube-nullity: there are tube-null sets of dimension 2 and sets of dimension 1 which are **not** tube-null. Still, intuitively, sets of large dimension should have more difficulty being tube-null.
- If we can decompose E into countably many pieces E_θ such that $P_\theta E_\theta$ is Lebesgue-null, then E is tube-null.
- There were very few non-trivial examples of tube-null sets of large dimension. In particular, it seems reasonable to ask which self-similar sets are tube-null.

Theorem (V. Harangi 2011)

The von Koch snowflake is tube-null.

Which sets are tube-null?

- There is no (non-trivial) connection between Hausdorff dimension and tube-nullity: there are tube-null sets of dimension 2 and sets of dimension 1 which are **not** tube-null. Still, intuitively, sets of large dimension should have more difficulty being tube-null.
- If we can decompose E into countably many pieces E_θ such that $P_\theta E_\theta$ is Lebesgue-null, then E is tube-null.
- There were very few non-trivial examples of tube-null sets of large dimension. In particular, it seems reasonable to ask which self-similar sets are tube-null.

Theorem (V. Harangi 2011)

The von Koch snowflake is tube-null.

Which sets are tube-null?

- There is no (non-trivial) connection between Hausdorff dimension and tube-nullity: there are tube-null sets of dimension 2 and sets of dimension 1 which are **not** tube-null. Still, intuitively, sets of large dimension should have more difficulty being tube-null.
- If we can decompose E into countably many pieces E_θ such that $P_\theta E_\theta$ is Lebesgue-null, then E is tube-null.
- There were very few non-trivial examples of tube-null sets of large dimension. In particular, it seems reasonable to ask which self-similar sets are tube-null.

Theorem (V. Harangi 2011)

The von Koch snowflake is tube-null.

The Sierpiński carpet is tube-null

Theorem (A. Pyörälä, P.S., V. Suomala and M. Wu 2020)

For any closed T_n-invariant set E, other than the full torus, there exists a finite set of rational directions θ_j and a decomposition $E = \cup_j E_j$ such that

$$\dim_{\mathrm{H}}(P_{\theta_j} E_j) < 1.$$

Corollary

Any non-trivial closed T_n-invariant set is tube null.

Corollary

The Sierpiński carpet is tube null.

The Sierpiński carpet is tube-null

Theorem (A. Pyörälä, P.S., V. Suomala and M. Wu 2020)

For any closed T_n-invariant set E, other than the full torus, there exists a finite set of rational directions θ_j and a decomposition $E = \cup_j E_j$ such that

$$\dim_H(P_{\theta_j} E_j) < 1.$$

Corollary

Any non-trivial closed T_n-invariant set is tube null.

Corollary

The Sierpiński carpet is tube null.

The Sierpiński carpet is tube-null

Theorem (A. Pyörälä, P.S., V. Suomala and M. Wu 2020)

For any closed T_n-invariant set E, other than the full torus, there exists a finite set of rational directions θ_j and a decomposition $E = \cup_j E_j$ such that

$$\dim_{\mathrm{H}}(P_{\theta_j} E_j) < 1.$$

Corollary

Any non-trivial closed T_n-invariant set is tube null.

Corollary

The Sierpiński carpet is tube null.

Some remarks on the result for the Sierpiński carpet

- Since the projection of the Sierpiński carpet in any direction is an interval, we need to decompose it into at least 2 pieces. By Baire's Theorem and self-similarity, the pieces can't be all closed (and none can be open).
- Our proof is indirect; we don't construct the pieces explicitly. (We can give an explicit set of directions that suffices.)
- The proof uses ergodic theory, in particular Bowen's Lemma relating topological entropy to measure-theoretic entropy.

Some remarks on the result for the Sierpiński carpet

- Since the projection of the Sierpiński carpet in any direction is an interval, we need to decompose it into at least 2 pieces. By Baire's Theorem and self-similarity, the pieces can't be all closed (and none can be open).
- Our proof is indirect; we don't construct the pieces explicitly. (We can give an explicit set of directions that suffices.)
- The proof uses ergodic theory, in particular Bowen's Lemma relating topological entropy to measure-theoretic entropy.

Some remarks on the result for the Sierpiński carpet

- Since the projection of the Sierpiński carpet in any direction is an interval, we need to decompose it into at least 2 pieces. By Baire's Theorem and self-similarity, the pieces can't be all closed (and none can be open).
- Our proof is indirect; we don't construct the pieces explicitly. (We can give an explicit set of directions that suffices.)
- The proof uses ergodic theory, in particular Bowen's Lemma relating topological entropy to measure-theoretic entropy.

A key proposition

Proposition (A.Pyörälä, P.S. , Ville Suomala, Meng Wu)

Let E be closed, T_n-invariant, and not the full torus. Then there are $c > 0$ and a ***finite*** *set Θ of rational directions, such that for every T_n-invariant measure μ supported on E there is $\theta \in \Theta$ such that*

$$\dim(P_\theta \mu) \leq 1 - c.$$

Corollary

Let

$$\mathcal{M}_\theta = \{\mu \in \mathcal{P}(E) : T_n\mu = \mu, \dim P_\theta\mu \leq 1 - c\}.$$

Then there exists a finite set of rational directions Θ such that

$$\mathcal{M} \subset \bigcup_{\theta\in\Theta} \mathcal{M}_\theta.$$

A key proposition

Proposition (A.Pyörälä, P.S. , Ville Suomala, Meng Wu)

*Let E be closed, T_n-invariant, and not the full torus. Then there are $c > 0$ and a **finite** set Θ of rational directions, such that for every T_n-invariant measure μ supported on E there is $\theta \in \Theta$ such that*

$$\dim(P_\theta \mu) \leq 1 - c.$$

Corollary

Let

$$\mathcal{M}_\theta = \{\mu \in \mathcal{P}(E) : T_n \mu = \mu, \dim P_\theta \mu \leq 1 - c\}.$$

Then there exists a finite set of rational directions Θ such that

$$\mathcal{M} \subset \bigcup_{\theta \in \Theta} \mathcal{M}_\theta.$$

The decomposition of E

Definition

Given $x \in E$, let $V(x)$ be the set of measures $\mu \in \mathcal{M}$ such that x is generic for μ along some subsequence or, in other words, the accumulation points of

$$\frac{1}{n}\sum_{j=0}^{n-1} \delta_{T_n^j x}.$$

Definition

$$E_\theta = \{x \in E : V(x) \cap \mathcal{M}_\theta \neq \varnothing\}.$$

Corollary (of key proposition)

$$E \subset \bigcup_{\theta \in \Theta} E_\theta.$$

The decomposition of E

Definition

Given $x \in E$, let $V(x)$ be the set of measures $\mu \in \mathcal{M}$ such that x is generic for μ along some subsequence or, in other words, the accumulation points of

$$\frac{1}{n}\sum_{j=0}^{n-1}\delta_{T_n^j x}.$$

Definition

$$E_\theta = \{x \in E : V(x) \cap \mathcal{M}_\theta \neq \varnothing\}.$$

Corollary (of key proposition)

$$E \subset \bigcup_{\theta\in\Theta} E_\theta.$$

The decomposition of E

Definition

Given $x \in E$, let $V(x)$ be the set of measures $\mu \in \mathcal{M}$ such that x is generic for μ along some subsequence or, in other words, the accumulation points of

$$\frac{1}{n}\sum_{j=0}^{n-1} \delta_{T_n^j x}.$$

Definition

$$E_\theta = \{x \in E : V(x) \cap \mathcal{M}_\theta \neq \varnothing\}.$$

Corollary (of key proposition)

$$E \subset \bigcup_{\theta \in \Theta} E_\theta.$$

Projections of T_n-invariant measures

Question (A. Algom)

Let μ be T_n-invariant and ergodic on $[0,1]^2$. When does there exist $\theta \notin \{0, \pi/2\}$ such that $\dim(P_\theta \mu) < \dim(\mu)$?

Corollary (A. Pyörälä, P.S., V.Suomala and M. Wu 2020)

Let μ be T_n-invariant and ergodic on $[0,1]^2$ and suppose $\dim \mu = 1$. Then the following are equivalent:

1. *$\mu = \nu \times \lambda$ or $\mu = \lambda \times \nu$, where λ is Lebesgue measure on $[0,1]$ and ν is a T_n-invariant measure of zero entropy.*
2. *$\dim(P_\theta \mu) = \dim \mu$ for all $\theta \notin \{0, \pi/2\}$.*

Projections of T_n-invariant measures

Question (A. Algom)

Let μ be T_n-invariant and ergodic on $[0,1]^2$. When does there exist $\theta \notin \{0, \pi/2\}$ such that $\dim(P_\theta \mu) < \dim(\mu)$?

Corollary (A. Pyörälä, P.S., V.Suomala and M. Wu 2020)

Let μ be T_n-invariant and ergodic on $[0,1]^2$ and suppose $\dim \mu = 1$. Then the following are equivalent:

1. *$\mu = \nu \times \lambda$ or $\mu = \lambda \times \nu$, where λ is Lebesgue measure on $[0,1]$ and ν is a T_n-invariant measure of zero entropy.*
2. *$\dim(P_\theta \mu) = \dim \mu$ for all $\theta \notin \{0, \pi/2\}$.*

Projections of T_n-invariant measures

Question (A. Algom)

Let μ be T_n-invariant and ergodic on $[0,1]^2$. When does there exist $\theta \notin \{0, \pi/2\}$ such that $\dim(P_\theta \mu) < \dim(\mu)$?

Corollary (A. Pyörälä, P.S., V.Suomala and M. Wu 2020)

Let μ be T_n-invariant and ergodic on $[0,1]^2$ and suppose $\dim \mu = 1$. Then the following are equivalent:

1. *$\mu = \nu \times \lambda$ or $\mu = \lambda \times \nu$, where λ is Lebesgue measure on $[0,1]$ and ν is a T_n-invariant measure of zero entropy.*
2. *$\dim(P_\theta \mu) = \dim \mu$ for all $\theta \notin \{0, \pi/2\}$.*

Projections of T_n-invariant measures

Question (A. Algom)

Let μ be T_n-invariant and ergodic on $[0,1]^2$. When does there exist $\theta \notin \{0, \pi/2\}$ such that $\dim(P_\theta \mu) < \dim(\mu)$?

Corollary (A. Pyörälä, P.S., V.Suomala and M. Wu 2020)

Let μ be T_n-invariant and ergodic on $[0,1]^2$ and suppose $\dim \mu = 1$. Then the following are equivalent:

1. $\mu = \nu \times \lambda$ *or* $\mu = \lambda \times \nu$, *where* λ *is Lebesgue measure on* $[0,1]$ *and* ν *is a* T_n*-invariant measure of zero entropy.*
2. $\dim(P_\theta \mu) = \dim \mu$ *for all* $\theta \notin \{0, \pi/2\}$.

Proof for the Sierpiński carpet: projected IFS

- The Sierpiński carpet K is the attractor of the IFS

$$\mathcal{F} = \left\{ f_{(i,j)} = \left(\frac{x+i}{3}, \frac{y+j}{3} \right) : (i,j) \in \Lambda \right\},$$

$$\Lambda = \{(0,0),(0,1),(0,2),(1,0),(1,2),(2,0),(2,1),(2,2)\}.$$

- Given $v \in \mathbb{R}^2 \setminus \{0\}$, let $P_v(x) = \langle v, x \rangle$; this is projection in direction v (scaled by $\|v\|$).
- Then $P_v K$ is the attractor of

$$\{\mathcal{F}_v = \tfrac{1}{3}(x + P_v(i,j)) : (i,j) \in \Lambda\}.$$

- In fact, $P_v K$ is an interval for all v so this is not too interesting. The projected IFS plays a crucial role but we have to look at projections of measures.

Proof for the Sierpiński carpet: projected IFS

- The Sierpiński carpet K is the attractor of the IFS

$$\mathcal{F} = \left\{ f_{(i,j)} = \left(\frac{x+i}{3}, \frac{y+j}{3} \right) : (i,j) \in \Lambda \right\},$$

$$\Lambda = \{(0,0),(0,1),(0,2),(1,0),(1,2),(2,0),(2,1),(2,2)\}.$$

- Given $v \in \mathbb{R}^2 \setminus \{0\}$, let $P_v(x) = \langle v, x \rangle$; this is projection in direction v (scaled by $\|v\|$).
- Then $P_v K$ is the attractor of

$$\{\mathcal{F}_v = \tfrac{1}{3}(x + P_v(i,j)) : (i,j) \in \Lambda\}.$$

- In fact, $P_v K$ is an interval for all v so this is not too interesting. The projected IFS plays a crucial role but we have to look at projections of measures.

Proof for the Sierpiński carpet: projected IFS

- The Sierpiński carpet K is the attractor of the IFS

$$\mathcal{F} = \left\{ f_{(i,j)} = \left(\frac{x+i}{3}, \frac{y+j}{3} \right) : (i,j) \in \Lambda \right\},$$

$$\Lambda = \{(0,0),(0,1),(0,2),(1,0),(1,2),(2,0),(2,1),(2,2)\}.$$

- Given $v \in \mathbb{R}^2 \setminus \{0\}$, let $P_v(x) = \langle v, x \rangle$; this is projection in direction v (scaled by $\|v\|$).
- Then $P_v K$ is the attractor of

$$\left\{ \mathcal{F}_v = \tfrac{1}{3}(x + P_v(i,j)) : (i,j) \in \Lambda \right\}.$$

- In fact, $P_v K$ is an interval for all v so this is not too interesting. The projected IFS plays a crucial role but we have to look at projections of measures.

Proof for the Sierpiński carpet: projected IFS

- The Sierpiński carpet K is the attractor of the IFS

$$\mathcal{F} = \left\{ f_{(i,j)} = \left(\frac{x+i}{3}, \frac{y+j}{3} \right) : (i,j) \in \Lambda \right\},$$

$$\Lambda = \{(0,0),(0,1),(0,2),(1,0),(1,2),(2,0),(2,1),(2,2)\}.$$

- Given $v \in \mathbb{R}^2 \setminus \{0\}$, let $P_v(x) = \langle v, x \rangle$; this is projection in direction v (scaled by $\|v\|$).
- Then $P_v K$ is the attractor of

$$\left\{ \mathcal{F}_v = \tfrac{1}{3}(x + P_v(i,j)) : (i,j) \in \Lambda \right\}.$$

- In fact, $P_v K$ is an interval for all v so this is not too interesting. The projected IFS plays a crucial role but we have to look at projections of measures.

Non-absolutely continuous projections

Let $\mathcal{M}$ be the collection of T_3-invariant measures supported on K.

Lemma

There is R_0 such that for every $\mu \in \mathcal{M}$ there is $v \in \mathbb{Z}^2 \cap B(0, R_0)$ such that $P_v\mu$ is not absolutely continuous.

Proof.

- Since μ is not Lebesgue, it has a non-zero Fourier coefficient, $\widehat{\mu}(p, q) \neq 0$, $(p, q) \neq (0, 0)$.
- Moreover, since Lebesgue is not in the weak closure of measures supported on K, we can find such (p, q) in a fixed ball of radius R_0.
- By T_3 invariance, this implies that if $v = (p, q)$, then

$$\widehat{P_v\mu}(3^n) = \widehat{\mu}(3^n p, 3^n q) = \widehat{\mu}(p, q) \neq 0.$$

- By the Riemann-Lebesgue Lemma, $P_v\mu \not\ll \mathcal{L}$.

□

Non-absolutely continuous projections

Let $\mathcal{M}$ be the collection of T_3-invariant measures supported on K.

Lemma

There is R_0 such that for every $\mu \in \mathcal{M}$ there is $v \in \mathbb{Z}^2 \cap B(0, R_0)$ such that $P_v\mu$ is not absolutely continuous.

Proof.

- Since μ is not Lebesgue, it has a non-zero Fourier coefficient, $\widehat{\mu}(p, q) \neq 0$, $(p, q) \neq (0, 0)$.
- Moreover, since Lebesgue is not in the weak closure of measures supported on K, we can find such (p, q) in a fixed ball of radius R_0.
- By T_3 invariance, this implies that if $v = (p, q)$, then

$$\widehat{P_v\mu}(3^n) = \widehat{\mu}(3^n p, 3^n q) = \widehat{\mu}(p, q) \neq 0.$$

- By the Riemann-Lebesgue Lemma, $P_v\mu \not\ll \mathcal{L}$.

□

Non-absolutely continuous projections

Let $\mathcal{M}$ be the collection of T_3-invariant measures supported on K.

Lemma

There is R_0 such that for every $\mu \in \mathcal{M}$ there is $v \in \mathbb{Z}^2 \cap B(0, R_0)$ such that $P_v\mu$ is not absolutely continuous.

Proof.

- Since μ is not Lebesgue, it has a non-zero Fourier coefficient, $\widehat{\mu}(p, q) \neq 0$, $(p, q) \neq (0, 0)$.
- Moreover, since Lebesgue is not in the weak closure of measures supported on K, we can find such (p, q) in a fixed ball of radius R_0.
- By T_3 invariance, this implies that if $v = (p, q)$, then

$$\widehat{P_v\mu}(3^n) = \widehat{\mu}(3^n p, 3^n q) = \widehat{\mu}(p, q) \neq 0.$$

- By the Riemann-Lebesgue Lemma, $P_v\mu \not\ll \mathcal{L}$.

□

Non-absolutely continuous projections

Let $\mathcal{M}$ be the collection of T_3-invariant measures supported on K.

Lemma

There is R_0 such that for every $\mu \in \mathcal{M}$ there is $v \in \mathbb{Z}^2 \cap B(0, R_0)$ such that $P_v\mu$ is not absolutely continuous.

Proof.

- Since μ is not Lebesgue, it has a non-zero Fourier coefficient, $\widehat{\mu}(p, q) \neq 0$, $(p, q) \neq (0, 0)$.
- Moreover, since Lebesgue is not in the weak closure of measures supported on K, we can find such (p, q) in a fixed ball of radius R_0.
- By T_3 invariance, this implies that if $v = (p, q)$, then

$$\widehat{P_v\mu}(3^n) = \widehat{\mu}(3^n p, 3^n q) = \widehat{\mu}(p, q) \neq 0.$$

- By the Riemann-Lebesgue Lemma, $P_v\mu \not\ll \mathcal{L}$.

□

Non-absolutely continuous projections

Let $\mathcal{M}$ be the collection of T_3-invariant measures supported on K.

Lemma

There is R_0 such that for every $\mu \in \mathcal{M}$ there is $v \in \mathbb{Z}^2 \cap B(0, R_0)$ such that $P_v\mu$ is not absolutely continuous.

Proof.

- Since μ is not Lebesgue, it has a non-zero Fourier coefficient, $\widehat{\mu}(p, q) \neq 0$, $(p, q) \neq (0, 0)$.
- Moreover, since Lebesgue is not in the weak closure of measures supported on K, we can find such (p, q) in a fixed ball of radius R_0.
- By T_3 invariance, this implies that if $v = (p, q)$, then

$$\widehat{P_v\mu}(3^n) = \widehat{\mu}(3^n p, 3^n q) = \widehat{\mu}(p, q) \neq 0.$$

- By the Riemann-Lebesgue Lemma, $P_v\mu \not\ll \mathcal{L}$.

□

Non-absolutely continuous projections

Let $\mathcal{M}$ be the collection of T_3-invariant measures supported on K.

Lemma

There is R_0 such that for every $\mu \in \mathcal{M}$ there is $v \in \mathbb{Z}^2 \cap B(0, R_0)$ such that $P_v\mu$ is not absolutely continuous.

Proof.

- Since μ is not Lebesgue, it has a non-zero Fourier coefficient, $\widehat{\mu}(p, q) \neq 0$, $(p, q) \neq (0, 0)$.
- Moreover, since Lebesgue is not in the weak closure of measures supported on K, we can find such (p, q) in a fixed ball of radius R_0.
- By T_3 invariance, this implies that if $v = (p, q)$, then

$$\widehat{P_v\mu}(3^n) = \widehat{\mu}(3^n p, 3^n q) = \widehat{\mu}(p, q) \neq 0.$$

- By the Riemann-Lebesgue Lemma, $P_v\mu \not\ll \mathcal{L}$.

□

Entropy dimension

Logarithms are to base 2

Definition (Entropy and entropy dimension)

- If μ is a measure and $\mathcal{A}$ is a measurable partition, we define the Shannon entropy

$$H(\mu, \mathcal{A}) = \sum_{A \in \mathcal{A}} \mu(A) \log(1/\mu(A)).$$

- If μ is a measure on $\mathbb{R}^d$, we define the entropy dimension as

$$\dim(\mu) = \lim_{n\to\infty} \frac{1}{n} H(\mu, \mathcal{D}_n(\mathbb{R}^d)) =: \lim_{n\to\infty} \frac{1}{n} H_n(\mu),$$

where $\mathcal{D}_n(\mathbb{R}^d)$ is the partition into dyadic 2^{-n}-cubes.

Entropy dimension

Logarithms are to base 2

Definition (Entropy and entropy dimension)

- If μ is a measure and $\mathcal{A}$ is a measurable partition, we define the Shannon entropy

$$H(\mu, \mathcal{A}) = \sum_{A \in \mathcal{A}} \mu(A) \log(1/\mu(A)).$$

- If μ is a measure on $\mathbb{R}^d$, we define the entropy dimension as

$$\dim(\mu) = \lim_{n\to\infty} \frac{1}{n} H(\mu, \mathcal{D}_n(\mathbb{R}^d)) =: \lim_{n\to\infty} \frac{1}{n} H_n(\mu),$$

where $\mathcal{D}_n(\mathbb{R}^d)$ is the partition into dyadic 2^{-n}-cubes.

Entropy dimension

Logarithms are to base 2

Definition (Entropy and entropy dimension)

- If μ is a measure and $\mathcal{A}$ is a measurable partition, we define the Shannon entropy

$$H(\mu, \mathcal{A}) = \sum_{A \in \mathcal{A}} \mu(A) \log(1/\mu(A)).$$

- If μ is a measure on $\mathbb{R}^d$, we define the entropy dimension as

$$\dim(\mu) = \lim_{n \to \infty} \frac{1}{n} H(\mu, \mathcal{D}_n(\mathbb{R}^d)) =: \lim_{n \to \infty} \frac{1}{n} H_n(\mu),$$

where $\mathcal{D}_n(\mathbb{R}^d)$ is the partition into dyadic 2^{-n}-cubes.

Basic properties of entropy dimension

- On $\mathbb{R}^d$, the entropy dimension ranges from 0 to d. Absolutely continuous measures have full entropy dimension.
- Hausdorff dimension $\le$ entropy dimension. This means that there are sets of positive μ-measure and Hausdorff dimension $\le \dim(\mu)$.
- If μ is T_n-invariant, then $\dim(\mu) = h(\mu, T_n)/\log n$.

Basic properties of entropy dimension

- On $\mathbb{R}^d$, the entropy dimension ranges from 0 to d. Absolutely continuous measures have full entropy dimension.
- Hausdorff dimension $\leq$ entropy dimension. This means that there are sets of positive μ-measure and Hausdorff dimension $\leq \dim(\mu)$.
- If μ is T_n-invariant, then $\dim(\mu) = h(\mu, T_n)/\log n$.

Basic properties of entropy dimension

- On $\mathbb{R}^d$, the entropy dimension ranges from 0 to d. Absolutely continuous measures have full entropy dimension.
- Hausdorff dimension $\leq$ entropy dimension. This means that there are sets of positive μ-measure and Hausdorff dimension $\leq \mathrm{dim}(\mu)$.
- If μ is T_n-invariant, then $\mathrm{dim}(\mu) = h(\mu, T_n)/\log n$.

Entropy of projected measures

Lemma

Let $v = (p, q) \in \mathbb{Z}^2 \setminus \{(0,0)\}$, and let $\mu \in \mathcal{M}$. Then

$$\textit{either} \quad P_v\mu \ll \mathcal{L} \quad \textit{or} \quad \dim P_v\mu < 1.$$

Moroever, $\mu \mapsto \dim P_v\mu$ is upper semicontinuous.

Proof.

- Show that

$$H_{n+m}(P_v\mu) \leq H_n(P_v\mu) + H_m(P_v\mu) + C_v.$$

  This holds because $\mathcal{F}_v$ satisfies the weak separation condition.
- This implies that if $\dim P_v\mu = 1$, then $H_n(P_v\mu) \geq n - C_v$.
- Any measure ν on $\mathbb{R}$ with $H_n(\nu) \geq n - C$ is absolutely continuous.

□

Entropy of projected measures

Lemma

Let $v = (p, q) \in \mathbb{Z}^2 \setminus \{(0,0)\}$, and let $\mu \in \mathcal{M}$. Then

$$\textit{either} \quad P_v\mu \ll \mathcal{L} \quad \textit{or} \quad \dim P_v\mu < 1.$$

Moroever, $\mu \mapsto \dim P_v\mu$ is upper semicontinuous.

Proof.

- Show that

$$H_{n+m}(P_v\mu) \leq H_n(P_v\mu) + H_m(P_v\mu) + C_v.$$

  This holds because $\mathcal{F}_v$ satisfies the weak separation condition.
- This implies that if $\dim P_v\mu = 1$, then $H_n(P_v\mu) \geq n - C_v$.
- Any measure ν on $\mathbb{R}$ with $H_n(\nu) \geq n - C$ is absolutely continuous.

□

Entropy of projected measures

Lemma

Let $v = (p, q) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$, *and let* $\mu \in \mathcal{M}$. *Then*

$$\textit{either} \quad P_v\mu \ll \mathcal{L} \quad \textit{or} \quad \dim P_v\mu < 1.$$

Moroever, $\mu \mapsto \dim P_v\mu$ *is upper semicontinuous.*

Proof.

- Show that

$$H_{n+m}(P_v\mu) \leq H_n(P_v\mu) + H_m(P_v\mu) + C_v.$$

This holds because $\mathcal{F}_v$ satisfies the weak separation condition.

- This implies that if $\dim P_v\mu = 1$, then $H_n(P_v\mu) \geq n - C_v$.
- Any measure ν on $\mathbb{R}$ with $H_n(\nu) \geq n - C$ is absolutely continuous.

□

Entropy of projected measures

Lemma

Let $v = (p,q) \in \mathbb{Z}^2 \setminus \{(0,0)\}$*, and let* $\mu \in \mathcal{M}$*. Then*

$$\textit{either} \quad P_v\mu \ll \mathcal{L} \quad \textit{or} \quad \dim P_v\mu < 1.$$

Moroever, $\mu \mapsto \dim P_v\mu$ *is upper semicontinuous.*

Proof.

- Show that

$$H_{n+m}(P_v\mu) \le H_n(P_v\mu) + H_m(P_v\mu) + C_v.$$

  This holds because $\mathcal{F}_v$ satisfies the weak separation condition.
- This implies that if $\dim P_v\mu = 1$, then $H_n(P_v\mu) \ge n - C_v$.
- Any measure ν on $\mathbb{R}$ with $H_n(\nu) \ge n - C$ is absolutely continuous.

□

Entropy of projected measures

Lemma

Let $v = (p, q) \in \mathbb{Z}^2 \setminus \{(0,0)\}$, *and let* $\mu \in \mathcal{M}$. *Then*

$$\textit{either} \quad P_v\mu \ll \mathcal{L} \quad \textit{or} \quad \dim P_v\mu < 1.$$

Moroever, $\mu \mapsto \dim P_v\mu$ *is upper semicontinuous.*

Proof.

- Show that

$$H_{n+m}(P_v\mu) \leq H_n(P_v\mu) + H_m(P_v\mu) + C_v.$$

  This holds because $\mathcal{F}_v$ satisfies the weak separation condition.
- This implies that if $\dim P_v\mu = 1$, then $H_n(P_v\mu) \geq n - C_v$.
- Any measure ν on $\mathbb{R}$ with $H_n(\nu) \geq n - C$ is absolutely continuous.

□

The weak separation condition

Definition

Let $(f_i)_{i=1}^m$ be an IFS. For each word $\mathtt{i} = (i_1 \ldots i_k) \in \{1, \ldots, m\}^k$, consider the composition

$$f_{\mathtt{i}} = f_{i_1} \circ \cdots \circ f_{i_k}.$$

The weak separation condition holds if any map of the form $f_{\mathtt{j}}^{-1} f_{\mathtt{i}}$, with $\mathtt{i}, \mathtt{j}$ words of the same length, is either equal to the identity or uniformly separated from the identity.

Remark

The weak separation condition allows for exact overlaps (that is, for coincidences $f_{\mathtt{i}} = f_{\mathtt{j}}$ for different words $\mathtt{i}, \mathtt{j}$), but it says that other than exact overlaps the pieces in the construction of the IFS are well separated.

The weak separation condition

Definition

Let $(f_i)_{i=1}^m$ be an IFS. For each word $\mathtt{i} = (i_1 \ldots i_k) \in \{1, \ldots, m\}^k$, consider the composition

$$f_{\mathtt{i}} = f_{i_1} \circ \cdots \circ f_{i_k}.$$

The weak separation condition holds if any map of the form $f_{\mathtt{j}}^{-1} f_{\mathtt{i}}$, with $\mathtt{i}, \mathtt{j}$ words of the same length, is either equal to the identity or uniformly separated from the identity.

Remark

The weak separation condition allows for exact overlaps (that is, for coincidences $f_{\mathtt{i}} = f_{\mathtt{j}}$ for different words $\mathtt{i}, \mathtt{j}$), but it says that other than exact overlaps the pieces in the construction of the IFS are well separated.

The key proposition

Putting everything together:

Proposition (A.Pyörälä, P.S. , Ville Suomala, Meng Wu)

Let

$$\mathcal{M}_\theta = \{\mu \in \mathcal{M} : \dim P_\theta \mu \le 1 - \delta_0\}.$$

Then there exists a finite set of rational directions Θ *such that*

$$\mathcal{M} \subset \bigcup_{\theta \in \Theta} \mathcal{M}_\theta.$$

Remark

It follows from a result of T. Jordan and A. Rapaport that if μ *is* T_n*-invariant,*

$$\dim(P_\theta \mu) = \min(\dim(\mu), 1)$$

for all irrational directions θ*.*

The key proposition

Putting everything together:

Proposition (A.Pyörälä, P.S. , Ville Suomala, Meng Wu)

Let

$$\mathcal{M}_\theta = \{\mu \in \mathcal{M} : \dim P_\theta \mu \leq 1 - \delta_0\}.$$

Then there exists a finite set of rational directions Θ *such that*

$$\mathcal{M} \subset \bigcup_{\theta \in \Theta} \mathcal{M}_\theta.$$

Remark

It follows from a result of T. Jordan and A. Rapaport that if μ *is* T_n*-invariant,*

$$\dim(P_\theta \mu) = \min(\dim(\mu), 1)$$

for all irrational directions θ*.*

The decomposition of K

Definition

Given $x \in K$, let $V(x)$ be the set of measures $\mu \in \mathcal{M}$ such that x is generic for μ along some subsequence or, in other words, the accumulation points of

$$\frac{1}{n}\sum_{j=0}^{n-1} \delta_{T_3^j x}.$$

Definition

$$K_\theta = \{x \in K : V(x) \cap \mathcal{M}_\theta \neq \varnothing\}.$$

Corollary (of key proposition)

$$K \subset \bigcup_{\theta \in \Theta} K_\theta.$$

The decomposition of K

Definition

Given $x \in K$, let $V(x)$ be the set of measures $\mu \in \mathcal{M}$ such that x is generic for μ along some subsequence or, in other words, the accumulation points of

$$\frac{1}{n}\sum_{j=0}^{n-1} \delta_{T_3^j x}.$$

Definition

$$K_\theta = \{x \in K : V(x) \cap \mathcal{M}_\theta \neq \varnothing\}.$$

Corollary (of key proposition)

$$K \subset \bigcup_{\theta \in \Theta} K_\theta.$$

The decomposition of K

Definition

Given $x \in K$, let $V(x)$ be the set of measures $\mu \in \mathcal{M}$ such that x is generic for μ along some subsequence or, in other words, the accumulation points of

$$\frac{1}{n}\sum_{j=0}^{n-1} \delta_{T_3^j x}.$$

Definition

$$K_\theta = \{x \in K : V(x) \cap \mathcal{M}_\theta \neq \varnothing\}.$$

Corollary (of key proposition)

$$K \subset \bigcup_{\theta \in \Theta} K_\theta.$$

The second key proposition

Corollary

$$K \subset \bigcup_{\theta \in \Theta} K_\theta .$$

To conclude the proof that the Sierpiński carpet is tube-null, it is enough to show:

Proposition

$$\dim_H(P_\theta K_\theta) < 1.$$

The second key proposition

Corollary

$$K \subset \bigcup_{\theta \in \Theta} K_\theta.$$

To conclude the proof that the Sierpiński carpet is tube-null, it is enough to show:

Proposition

$$\dim_{\mathrm{H}}(P_\theta K_\theta) < 1.$$

Identifying exact overlaps

- Fix $\theta = (p, q) \in \Theta$ and $\mu \in \mathcal{M}_\theta$. Recall that this means that $\dim P_\theta \mu \leq 1 - \delta_0$.
- We replace the projected IFS $\mathcal{F}_v$ by a sufficiently high iteration $\mathcal{F}_v^k = \{P_v f_{\mathtt{i}} : \mathtt{i} \in \Lambda^k\}$.
- Many of the maps $P_v f_{\mathtt{i}}$ coincide. We consider the factor map $\pi = \pi_v$ that identifies all words $\mathtt{i} \in \Lambda^k$ according to the equivalence relation $P_v f_{\mathtt{i}} = P_v f_{\mathtt{j}}$.

Lemma

If k is large enough and $\mu \in \mathcal{M}_\theta$,

$$h(\pi\mu, \sigma) \leq (1 - \delta_0/2) \log 3.$$

Proof.

By the WSC, if k is large then, after identifying exact overlaps, the map π_V is "almost injective". □

Identifying exact overlaps

- Fix $\theta = (p, q) \in \Theta$ and $\mu \in \mathcal{M}_\theta$. Recall that this means that $\dim P_\theta \mu \leq 1 - \delta_0$.
- We replace the projected IFS $\mathcal{F}_v$ by a sufficiently high iteration $\mathcal{F}_v^k = \{P_v f_{\mathtt{i}} : \mathtt{i} \in \Lambda^k\}$.
- Many of the maps $P_v f_{\mathtt{i}}$ coincide. We consider the factor map $\pi = \pi_v$ that identifies all words $\mathtt{i} \in \Lambda^k$ according to the equivalence relation $P_v f_{\mathtt{i}} = P_v f_{\mathtt{j}}$.

Lemma

If k is large enough and $\mu \in \mathcal{M}_\theta$,

$$h(\pi\mu, \sigma) \leq (1 - \delta_0/2) \log 3.$$

Proof.

By the WSC, if k is large then, after identifying exact overlaps, the map π_v is "almost injective". □

Identifying exact overlaps

- Fix $\theta = (p, q) \in \Theta$ and $\mu \in \mathcal{M}_\theta$. Recall that this means that $\dim P_\theta \mu \le 1 - \delta_0$.
- We replace the projected IFS $\mathcal{F}_v$ by a sufficiently high iteration $\mathcal{F}_v^k = \{P_v f_{\mathtt{i}} : \mathtt{i} \in \Lambda^k\}$.
- Many of the maps $P_v f_{\mathtt{i}}$ coincide. We consider the factor map $\pi = \pi_v$ that identifies all words $\mathtt{i} \in \Lambda^k$ according to the equivalence relation $P_v f_{\mathtt{i}} = P_v f_{\mathtt{j}}$.

Lemma

If k is large enough and $\mu \in \mathcal{M}_\theta$,

$$h(\pi\mu, \sigma) \le (1 - \delta_0/2) \log 3.$$

Proof.

By the WSC, if k is large then, after identifying exact overlaps, the map π_v is "almost injective". □

Identifying exact overlaps

- Fix $\theta = (p, q) \in \Theta$ and $\mu \in \mathcal{M}_\theta$. Recall that this means that $\dim P_\theta \mu \leq 1 - \delta_0$.
- We replace the projected IFS $\mathcal{F}_v$ by a sufficiently high iteration $\mathcal{F}_v^k = \{P_v f_{\mathtt{i}} : \mathtt{i} \in \Lambda^k\}$.
- Many of the maps $P_v f_{\mathtt{i}}$ coincide. We consider the factor map $\pi = \pi_v$ that identifies all words $\mathtt{i} \in \Lambda^k$ according to the equivalence relation $P_v f_{\mathtt{i}} = P_v f_{\mathtt{j}}$.

Lemma

If k is large enough and $\mu \in \mathcal{M}_\theta$,

$$h(\pi\mu, \sigma) \leq (1 - \delta_0/2) \log 3.$$

Proof.

By the WSC, if k is large then, after identifying exact overlaps, the map π_v is "almost injective". □

Identifying exact overlaps

- Fix $\theta = (p, q) \in \Theta$ and $\mu \in \mathcal{M}_\theta$. Recall that this means that $\dim P_\theta \mu \leq 1 - \delta_0$.
- We replace the projected IFS $\mathcal{F}_v$ by a sufficiently high iteration $\mathcal{F}_v^k = \{P_v f_{\mathtt{i}} : \mathtt{i} \in \Lambda^k\}$.
- Many of the maps $P_v f_{\mathtt{i}}$ coincide. We consider the factor map $\pi = \pi_v$ that identifies all words $\mathtt{i} \in \Lambda^k$ according to the equivalence relation $P_v f_{\mathtt{i}} = P_v f_{\mathtt{j}}$.

Lemma

If k is large enough and $\mu \in \mathcal{M}_\theta$,

$$h(\pi\mu, \sigma) \leq (1 - \delta_0/2) \log 3.$$

Proof.

By the WSC, if k is large then, after identifying exact overlaps, the map π_v is "almost injective". □

Conclusion of the proof

- K_θ =points that equidistribute (along some subsequence) for some $\mu \in \mathcal{M}_\theta$.
- If π is "identifying the overlaps of high iteration" then $h(\pi\mu, \sigma) \leq (1 - \delta_0/2) \log 3$ for $\mu \in \mathcal{M}_\theta$.
- If x equidistributes for μ, then πx equidistributes for $\pi\mu$.
- Therefore if $x \in K_\theta$, then πx equidistributes for some measure of entropy $\leq (1 - \delta_0/2) \log 3$.
- The proof is now concluded from Bowen's Lemma.

Lemma (Bowen)

If E_t is the set of points in $\Gamma^{\mathbb{N}}$ that equidistribute (under some subsequence) for some measure of entropy $\leq t$, then

$$h_{\text{top}}(E_t, \sigma) \leq t.$$

Conclusion of the proof

- K_θ =points that equidistribute (along some subsequence) for some $\mu \in \mathcal{M}_\theta$.
- If π is "identifying the overlaps of high iteration" then $h(\pi\mu, \sigma) \leq (1 - \delta_0/2) \log 3$ for $\mu \in \mathcal{M}_\theta$.
- If x equidistributes for μ, then πx equidistributes for $\pi\mu$.
- Therefore if $x \in K_\theta$, then πx equidistributes for some measure of entropy $\leq (1 - \delta_0/2) \log 3$.
- The proof is now concluded from Bowen's Lemma.

Lemma (Bowen)

If E_t is the set of points in $\Gamma^{\mathbb{N}}$ that equidistribute (under some subsequence) for some measure of entropy $\leq t$, then

$$h_{\text{top}}(E_t, \sigma) \leq t.$$

Conclusion of the proof

- K_θ =points that equidistribute (along some subsequence) for some $\mu \in \mathcal{M}_\theta$.
- If π is "identifying the overlaps of high iteration" then $h(\pi\mu, \sigma) \leq (1 - \delta_0/2)\log 3$ for $\mu \in \mathcal{M}_\theta$.
- If x equidistributes for μ, then πx equidistributes for $\pi\mu$.
- Therefore if $x \in K_\theta$, then πx equidistributes for some measure of entropy $\leq (1 - \delta_0/2)\log 3$.
- The proof is now concluded from Bowen's Lemma.

Lemma (Bowen)

If E_t is the set of points in $\Gamma^{\mathbb{N}}$ that equidistribute (under some subsequence) for some measure of entropy $\leq t$, then

$$h_{\text{top}}(E_t, \sigma) \leq t.$$

Conclusion of the proof

- K_θ =points that equidistribute (along some subsequence) for some $\mu \in \mathcal{M}_\theta$.
- If π is "identifying the overlaps of high iteration" then $h(\pi\mu, \sigma) \leq (1 - \delta_0/2) \log 3$ for $\mu \in \mathcal{M}_\theta$.
- If x equidistributes for μ, then πx equidistributes for $\pi\mu$.
- Therefore if $x \in K_\theta$, then πx equidistributes for some measure of entropy $\leq (1 - \delta_0/2) \log 3$.
- The proof is now concluded from Bowen's Lemma.

Lemma (Bowen)

If E_t is the set of points in $\Gamma^{\mathbb{N}}$ that equidistribute (under some subsequence) for some measure of entropy $\leq t$, then

$$h_{\text{top}}(E_t, \sigma) \leq t.$$

Conclusion of the proof

- K_θ =points that equidistribute (along some subsequence) for some $\mu \in \mathcal{M}_\theta$.
- If π is "identifying the overlaps of high iteration" then $h(\pi\mu, \sigma) \leq (1 - \delta_0/2) \log 3$ for $\mu \in \mathcal{M}_\theta$.
- If x equidistributes for μ, then πx equidistributes for $\pi\mu$.
- Therefore if $x \in K_\theta$, then πx equidistributes for some measure of entropy $\leq (1 - \delta_0/2) \log 3$.
- The proof is now concluded from Bowen's Lemma.

Lemma (Bowen)

If E_t is the set of points in $\Gamma^{\mathbb{N}}$ that equidistribute (under some subsequence) for some measure of entropy $\leq t$, then

$$h_{\text{top}}(E_t, \sigma) \leq t.$$

Conclusion of the proof

- K_θ =points that equidistribute (along some subsequence) for some $\mu \in \mathcal{M}_\theta$.
- If π is "identifying the overlaps of high iteration" then $h(\pi\mu, \sigma) \leq (1 - \delta_0/2)\log 3$ for $\mu \in \mathcal{M}_\theta$.
- If x equidistributes for μ, then πx equidistributes for $\pi\mu$.
- Therefore if $x \in K_\theta$, then πx equidistributes for some measure of entropy $\leq (1 - \delta_0/2)\log 3$.
- The proof is now concluded from Bowen's Lemma.

Lemma (Bowen)

If E_t is the set of points in $\Gamma^{\mathbb{N}}$ that equidistribute (under some subsequence) for some measure of entropy $\leq t$, then

$$h_{\text{top}}(E_t, \sigma) \leq t.$$

Other results: self-similar sets with no rotations

Question

We have seen that carpet-type self-similar sets are tube-null. What about other self-similar sets?

Theorem (A. Pyörälä, P.S., V. Suomala and M. Wu)

Let $\{rx + t_i\}_{i=1}^4$ be a homogeneous IFS with 4 maps and no rotations, and let K be the attractor.

If $r < 2^{-3/2} \approx 0.353$, and $\Theta = \{t_i - t_j : i \neq j\}$, there are sets $(K_\theta)_{\theta\in\Theta}$ covering K such that $\dim(P_\theta K_\theta) < 1$.

In particular, K is tube-null.

Other results: self-similar sets with no rotations

Question

We have seen that carpet-type self-similar sets are tube-null. What about other self-similar sets?

Theorem (A. Pyörälä, P.S., V. Suomala and M. Wu)

Let $\{rx + t_i\}_{i=1}^4$ be a homogeneous IFS with 4 maps and no rotations, and let K be the attractor.

If $r < 2^{-3/2} \approx 0.353$, and $\Theta = \{t_i - t_j : i \neq j\}$, there are sets $(K_\theta)_{\theta \in \Theta}$ covering K such that $\dim(P_\theta K_\theta) < 1$.

In particular, K is tube-null.

A tube-null, non-carpet self-similar set

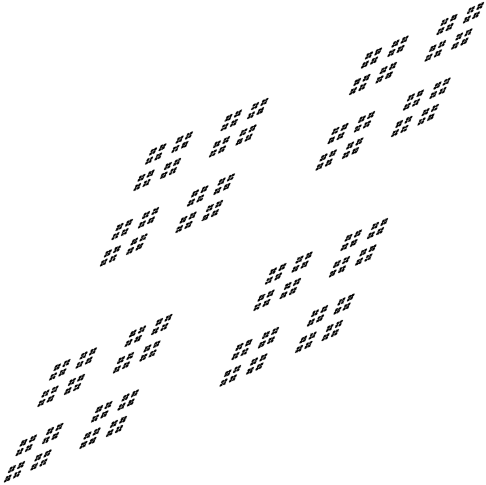


Figure: A self-similar set of dimension ≈ 1.3205. It is tube-null, even though it can be checked that all its projections are intervals

Remarks on self-similar sets without rotations

Theorem

If K is a homogeneous self-similar sets with no rotations, 4 maps and contraction ratio $< 2^{-3/2} \approx 0.353$, then K is tube null.

- If K satisfies OSC, the condition is equivalent to $\dim_H(K) < 4/3$.
- If $r < 1/3$ (equivalently $\dim_H(K) < 1.2618\ldots$), the result is almost trivial: for any direction in Θ, the projection of all of K has $\dim_H < 1$.
- On the other hand, if $r > 1/3$, as we have seen this is not true: the projections of K in all directions may be intervals. We use a similar argument to the carpet case (but easier).
- Similar results hold for any number of maps and non-homogeneous IFS's. But it is key that there are no rotations.

Remarks on self-similar sets without rotations

Theorem

If K is a homogeneous self-similar sets with no rotations, 4 maps and contraction ratio $< 2^{-3/2} \approx 0.353$, then K is tube null.

- If K satisfies OSC, the condition is equivalent to $\dim_H(K) < 4/3$.
- If $r < 1/3$ (equivalently $\dim_H(K) < 1.2618\ldots$), the result is almost trivial: for any direction in Θ, the projection of all of K has $\dim_H < 1$.
- On the other hand, if $r > 1/3$, as we have seen this is not true: the projections of K in all directions may be intervals. We use a similar argument to the carpet case (but easier).
- Similar results hold for any number of maps and non-homogeneous IFS's. But it is key that there are no rotations.

Remarks on self-similar sets without rotations

Theorem

If K is a homogeneous self-similar sets with no rotations, 4 maps and contraction ratio $< 2^{-3/2} \approx 0.353$, then K is tube null.

- If K satisfies OSC, the condition is equivalent to $\dim_H(K) < 4/3$.
- If $r < 1/3$ (equivalently $\dim_H(K) < 1.2618\ldots$), the result is almost trivial: for any direction in Θ, the projection of all of K has $\dim_H < 1$.
- On the other hand, if $r > 1/3$, as we have seen this is not true: the projections of K in all directions may be intervals. We use a similar argument to the carpet case (but easier).
- Similar results hold for any number of maps and non-homogeneous IFS's. But it is key that there are no rotations.

Remarks on self-similar sets without rotations

Theorem

If K is a homogeneous self-similar sets with no rotations, 4 maps and contraction ratio $< 2^{-3/2} \approx 0.353$, then K is tube null.

- If K satisfies OSC, the condition is equivalent to $\dim_H(K) < 4/3$.
- If $r < 1/3$ (equivalently $\dim_H(K) < 1.2618\ldots$), the result is almost trivial: for any direction in Θ, the projection of all of K has $\dim_H < 1$.
- On the other hand, if $r > 1/3$, as we have seen this is not true: the projections of K in all directions may be intervals. We use a similar argument to the carpet case (but easier).
- Similar results hold for any number of maps and non-homogeneous IFS's. But it is key that there are no rotations.

Remarks on self-similar sets without rotations

Theorem

If K is a homogeneous self-similar sets with no rotations, 4 maps and contraction ratio $< 2^{-3/2} \approx 0.353$, then K is tube null.

- If K satisfies OSC, the condition is equivalent to $\dim_H(K) < 4/3$.
- If $r < 1/3$ (equivalently $\dim_H(K) < 1.2618\ldots$), the result is almost trivial: for any direction in Θ, the projection of all of K has $\dim_H < 1$.
- On the other hand, if $r > 1/3$, as we have seen this is not true: the projections of K in all directions may be intervals. We use a similar argument to the carpet case (but easier).
- Similar results hold for any number of maps and non-homogeneous IFS's. But it is key that there are no rotations.

Other results: self-similar sets with dense rotations

Theorem (A. Pyörälä, P.S., V. Suomala and M. Wu)

Let $\{f_i(x) = \lambda_i R_{\theta_i} x + t_i\}$ be a self-similar IFS, where R_θ is rotation by angle θ, and let K be the attractor.

If $\dim_H(K) \geq 1$ and there is θ with $\theta/\pi \notin \mathbb{Q}$ ("dense rotations"), then for every $\delta > 0$ there is $c = c_\delta > 0$ such that for any covering $(T_j)_j$ of K by tubes,

$$\sum_j w(T_j)^{1-\delta} \geq c > 0.$$

Remark

If we define a "tube Hausdorff dimension" using covering by tubes and $w(T)$ instead of the diameter, the theorem says that self-similar sets with dense rotation of dimension ≥ 1 have tube Hausdorff dimension equal to 1 (maximum possible value).

Other results: self-similar sets with dense rotations

Theorem (A. Pyörälä, P.S., V. Suomala and M. Wu)

Let $\{f_i(x) = \lambda_i R_{\theta_i} x + t_i\}$ *be a self-similar IFS, where* R_θ *is rotation by angle* θ*, and let* K *be the attractor.*

If $\dim_H(K) \geq 1$ *and there is* θ *with* $\theta/\pi \notin \mathbb{Q}$ *("dense rotations"), then for every* $\delta > 0$ *there is* $c = c_\delta > 0$ *such that for any covering* $(T_j)_j$ *of* K *by tubes,*

$$\sum_j w(T_j)^{1-\delta} \geq c > 0.$$

Remark

If we define a "tube Hausdorff dimension" using covering by tubes and $w(T)$ *instead of the diameter, the theorem says that self-similar sets with dense rotation of dimension* ≥ 1 *have tube Hausdorff dimension equal to* 1 *(maximum possible value).*

Remarks on self-similar sets with dense rotations

Theorem

Self-similar sets in the plane with dense rotations and dimension ≥ 1 have "tube Hausdorff dimension" 1.

- We believe that such self-similar sets are **not** tube-null, but this seems to be very difficult to prove. What we prove is just slightly weaker.
- Our proof for Sierpiński carpets shows that they have tube dimension < 1, so there is definitely a contrast.
- By a rather standard reduction, it is enough to consider homogeneous self-similar sets with strong separation. Then the result is a consequence of the slicing results from the first part of the talk.

Remarks on self-similar sets with dense rotations

Theorem

Self-similar sets in the plane with dense rotations and dimension ≥ 1 have "tube Hausdorff dimension" 1.

- We believe that such self-similar sets are **not** tube-null, but this seems to be very difficult to prove. What we prove is just slightly weaker.
- Our proof for Sierpiński carpets shows that they have tube dimension < 1, so there is definitely a contrast.
- By a rather standard reduction, it is enough to consider homogeneous self-similar sets with strong separation. Then the result is a consequence of the slicing results from the first part of the talk.

Remarks on self-similar sets with dense rotations

Theorem

Self-similar sets in the plane with dense rotations and dimension ≥ 1 have "tube Hausdorff dimension" 1.

- We believe that such self-similar sets are **not** tube-null, but this seems to be very difficult to prove. What we prove is just slightly weaker.
- Our proof for Sierpiński carpets shows that they have tube dimension < 1, so there is definitely a contrast.
- By a rather standard reduction, it is enough to consider homogeneous self-similar sets with strong separation. Then the result is a consequence of the slicing results from the first part of the talk.

Remarks on self-similar sets with dense rotations

Theorem

Self-similar sets in the plane with dense rotations and dimension ≥ 1 have "tube Hausdorff dimension" 1.

- We believe that such self-similar sets are **not** tube-null, but this seems to be very difficult to prove. What we prove is just slightly weaker.
- Our proof for Sierpiński carpets shows that they have tube dimension < 1, so there is definitely a contrast.
- By a rather standard reduction, it is enough to consider homogeneous self-similar sets with strong separation. Then the result is a consequence of the slicing results from the first part of the talk.

Thank you!!