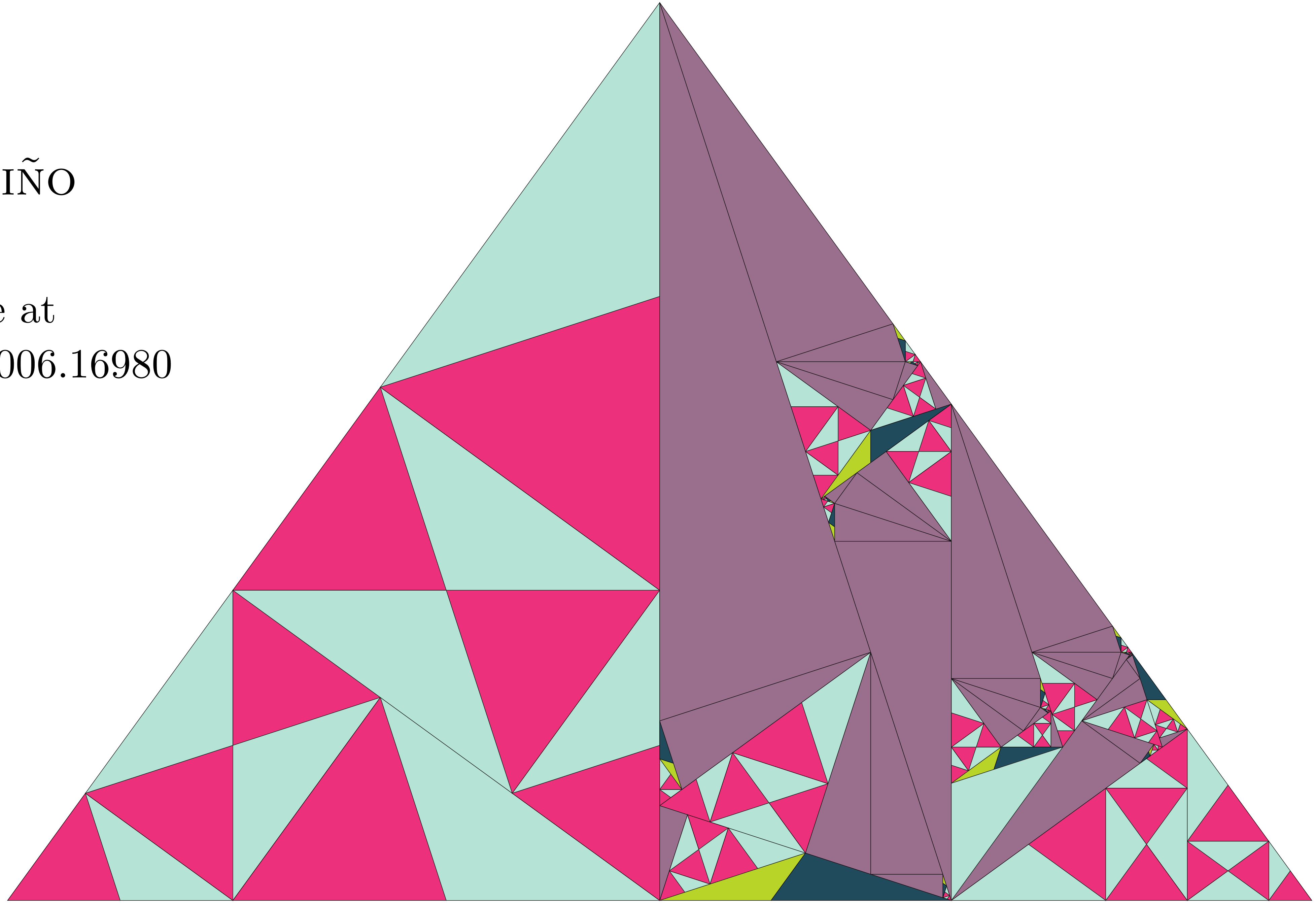
Quantitative weak mixing for random substitution tilings

Rodrigo Treviño

Preprint available at
https://arxiv.org/abs/2006.16980



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Quantitative weak mixing is about bounding the dimension of spectral measures from below (away from 0).

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Proposition (Hof): If $|\mathcal{S}_R^x(f,\lambda)| \leq CR^{d-\alpha}$ for any x and $R > R_0$, then

$$\mu_f(B_r(\lambda)) \leq C' r^{2\alpha}$$

for all r small enough, which implies that

$$d^-_{\mu_f}(\lambda) \geq 2\alpha.$$

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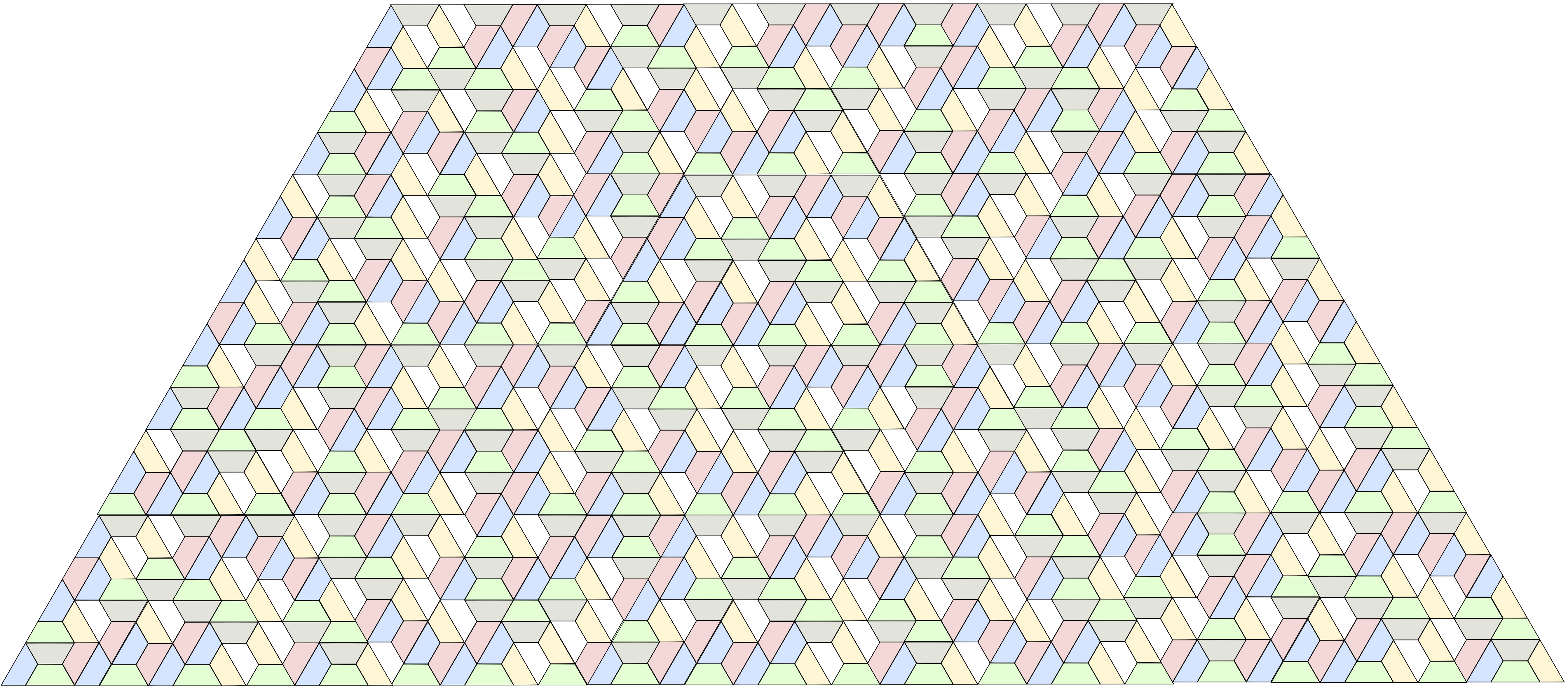

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The Čech cohomology is

$$\check{H}^*(\Omega;\mathbb{Z}) = \varinjlim\,(H^*(\Gamma_k;\mathbb{Z}), \gamma_k^*)$$

Random substitution tilings

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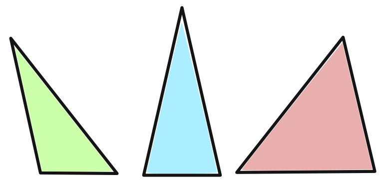

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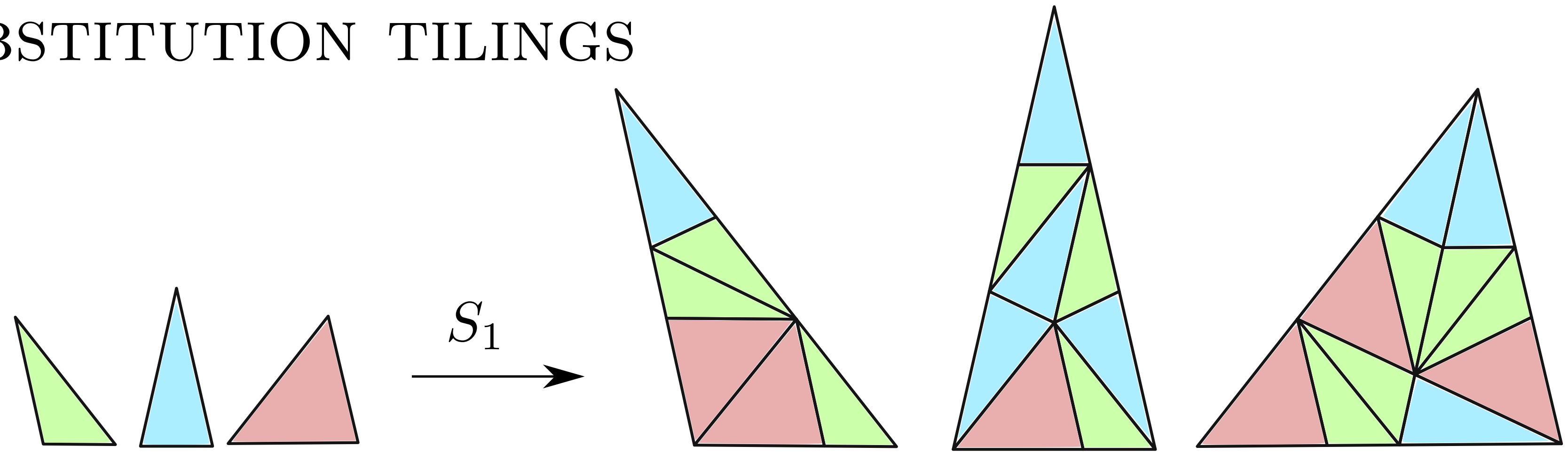

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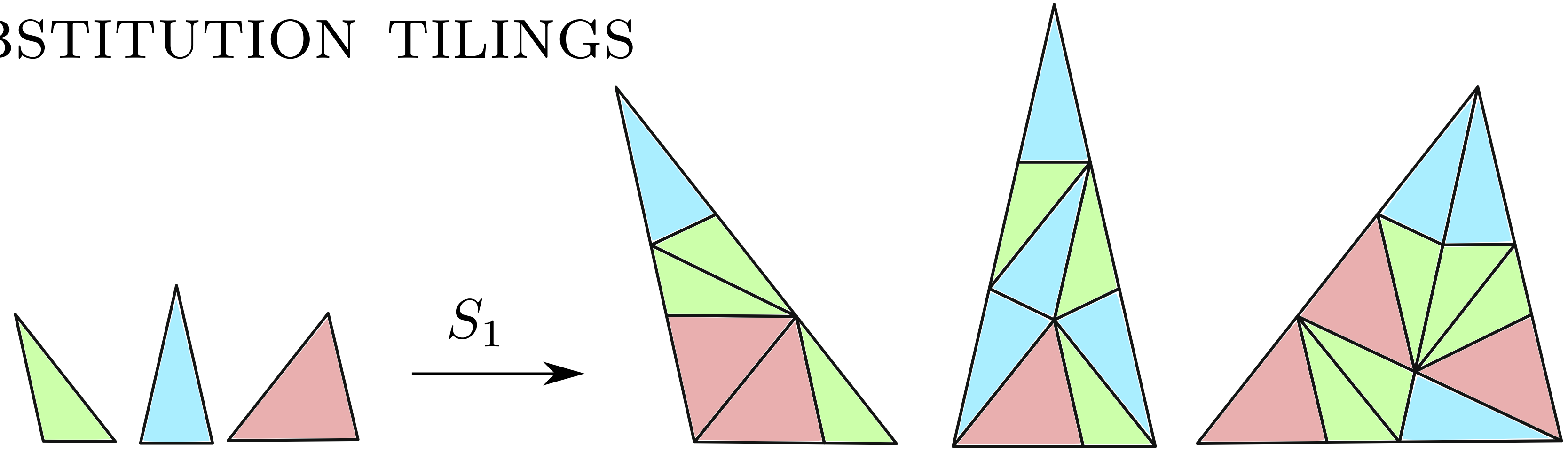

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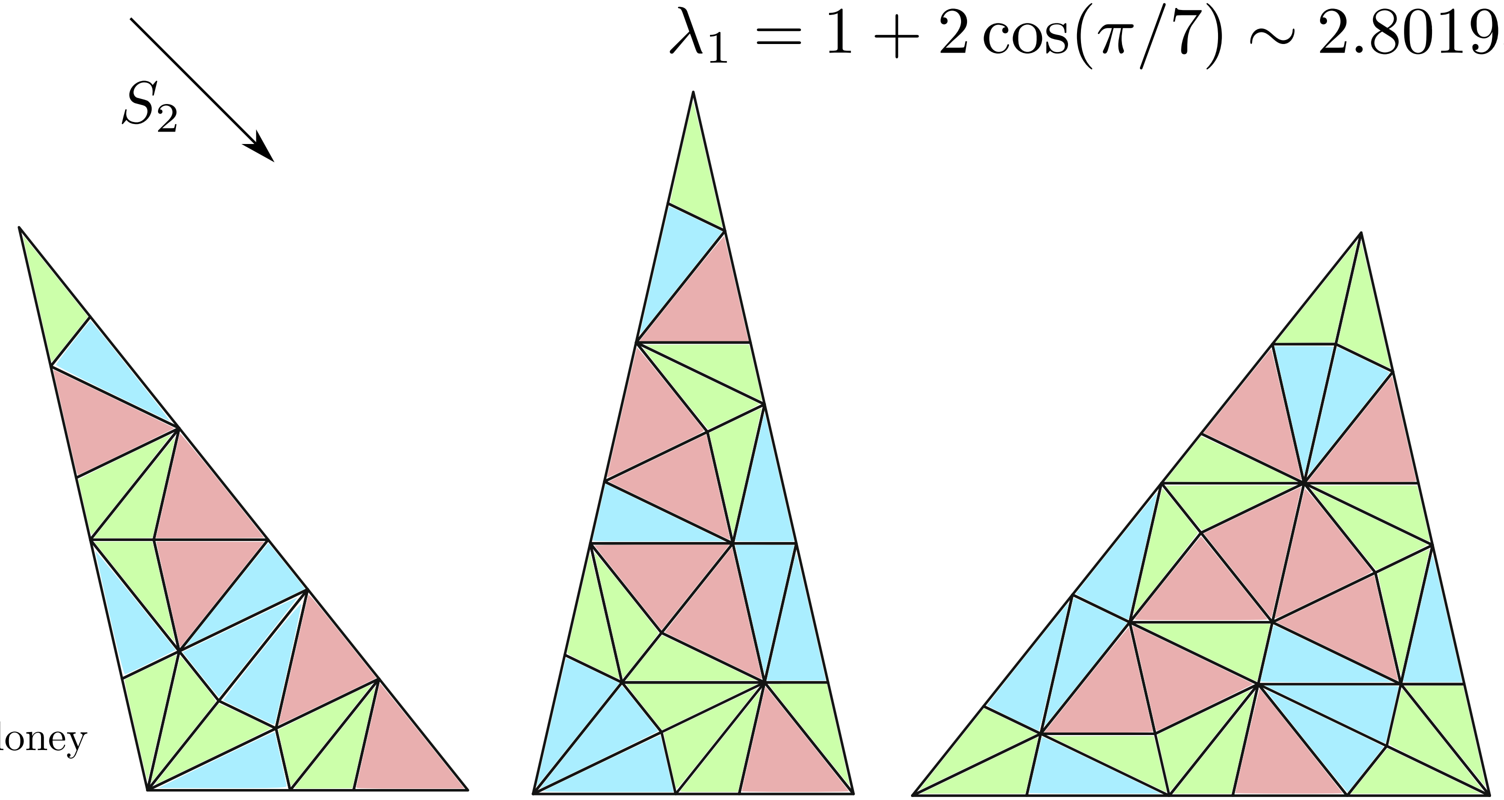

Gähler, Kwan, Maloney

$$\lambda_2 = 2\cos(\pi/7)(1 + 2\cos(\pi/7)) - 1 \sim 4.0489...$$

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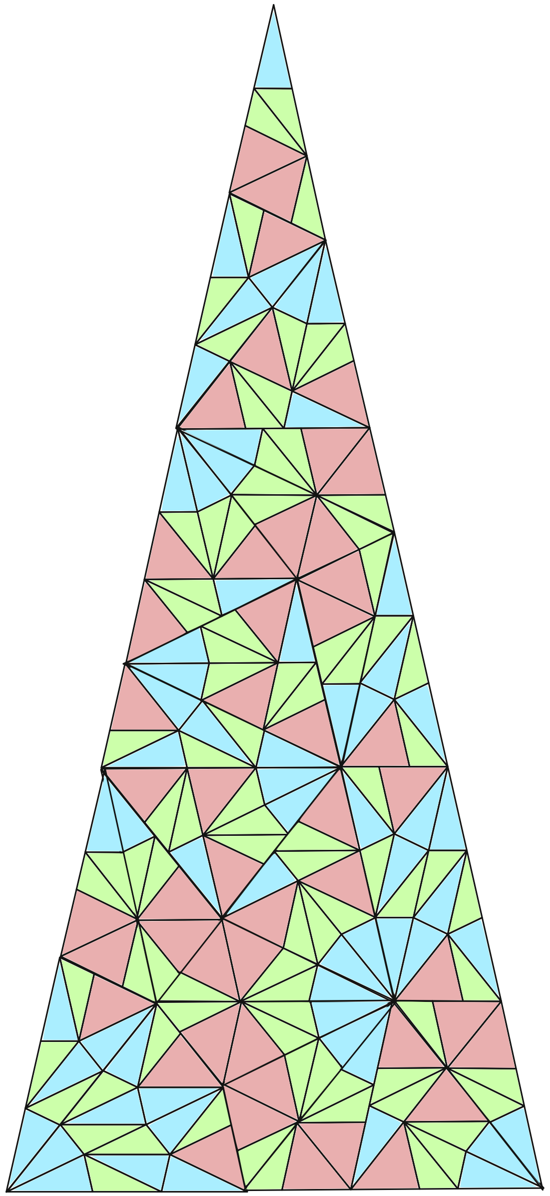

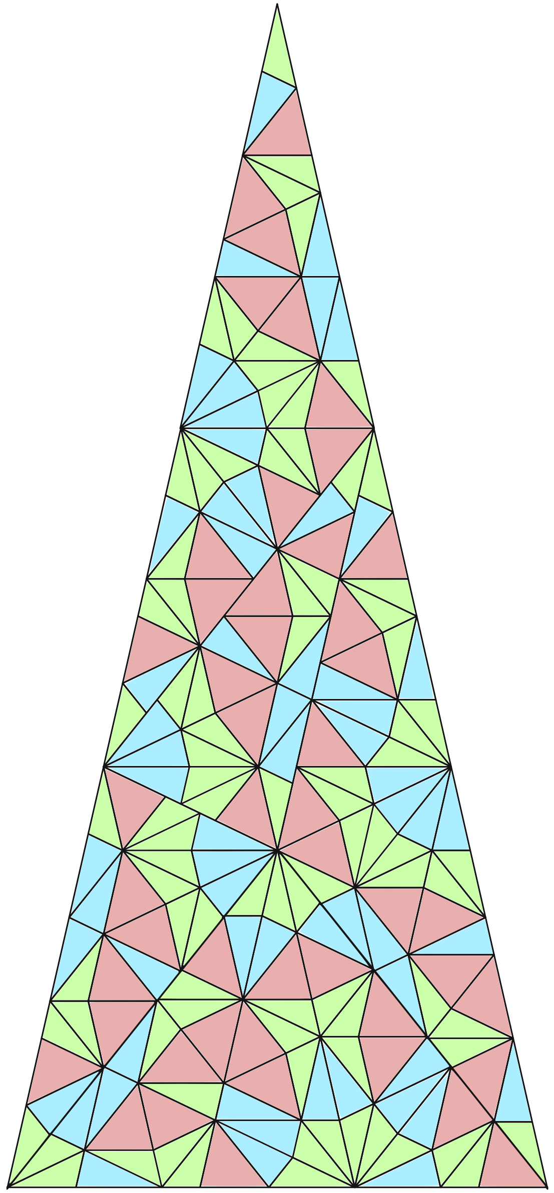

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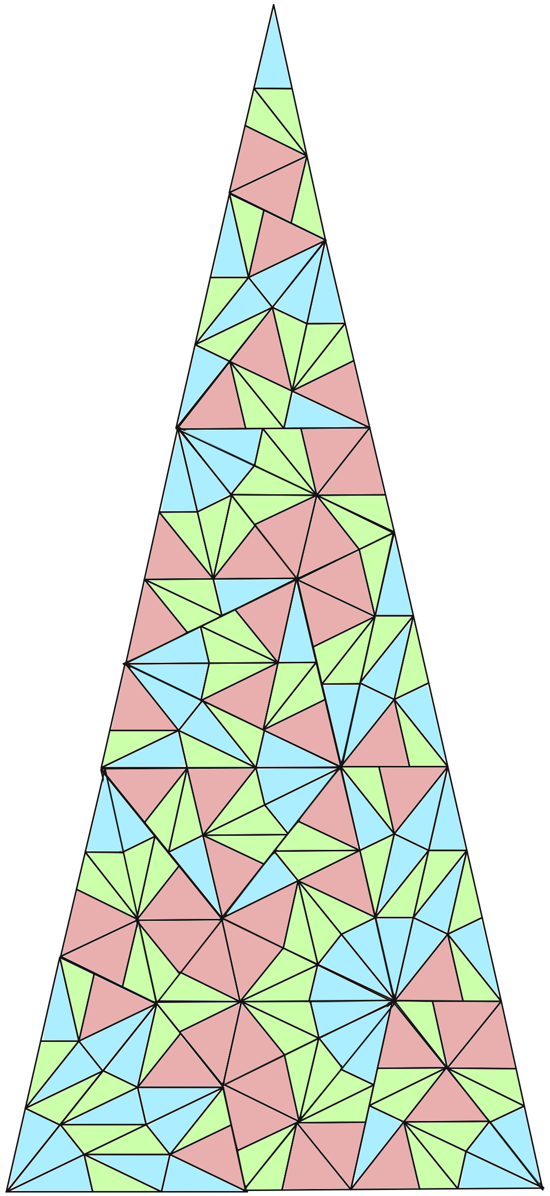
Given $x \in \Sigma_2$, we construct a tiling space Ω_x



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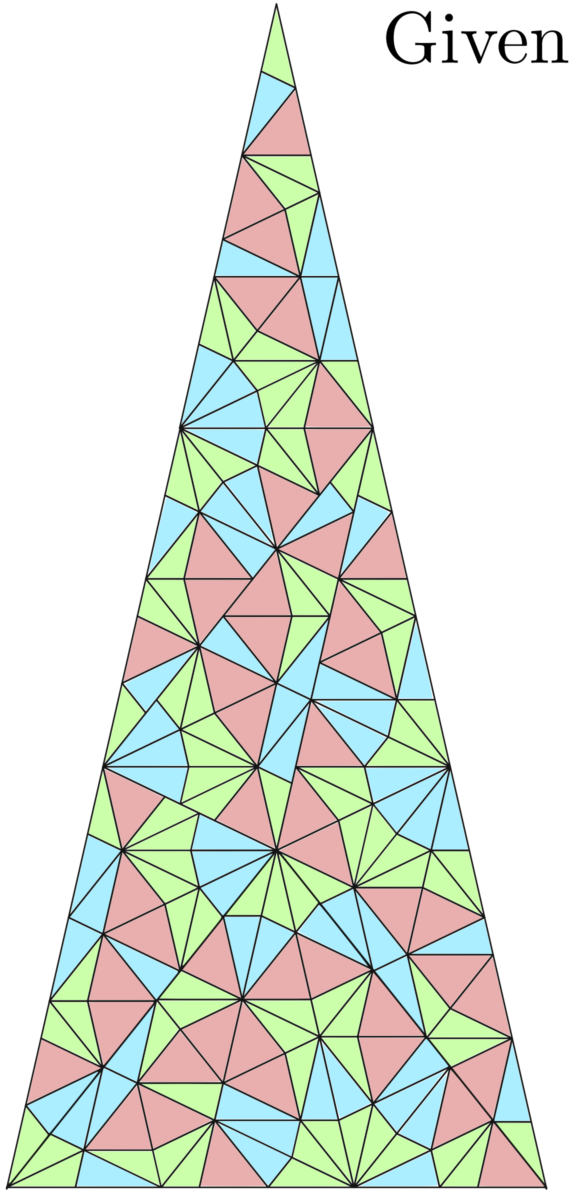

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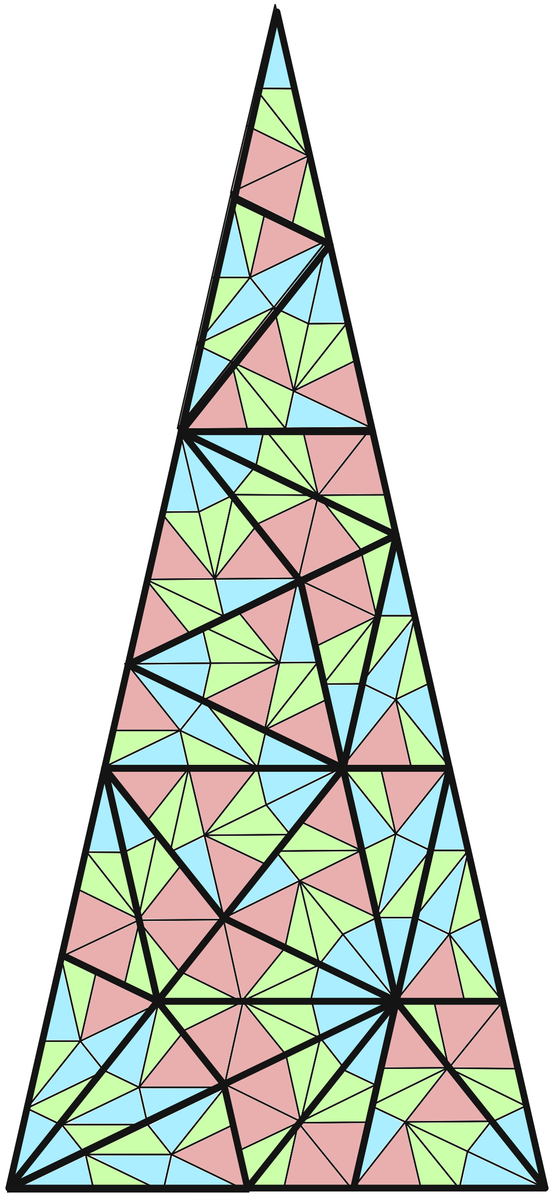

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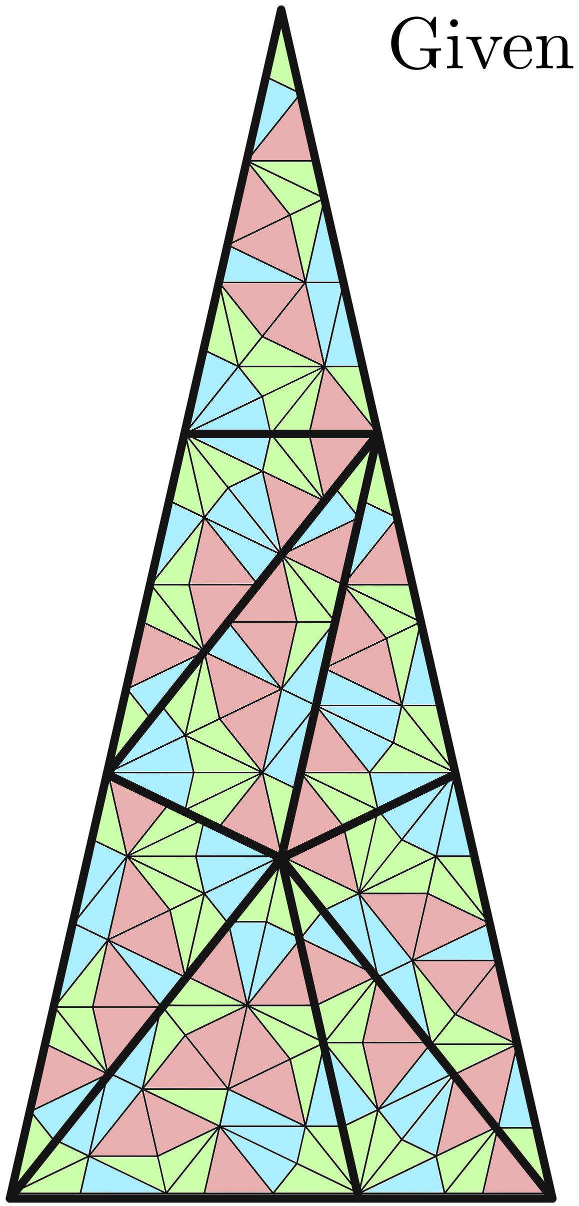

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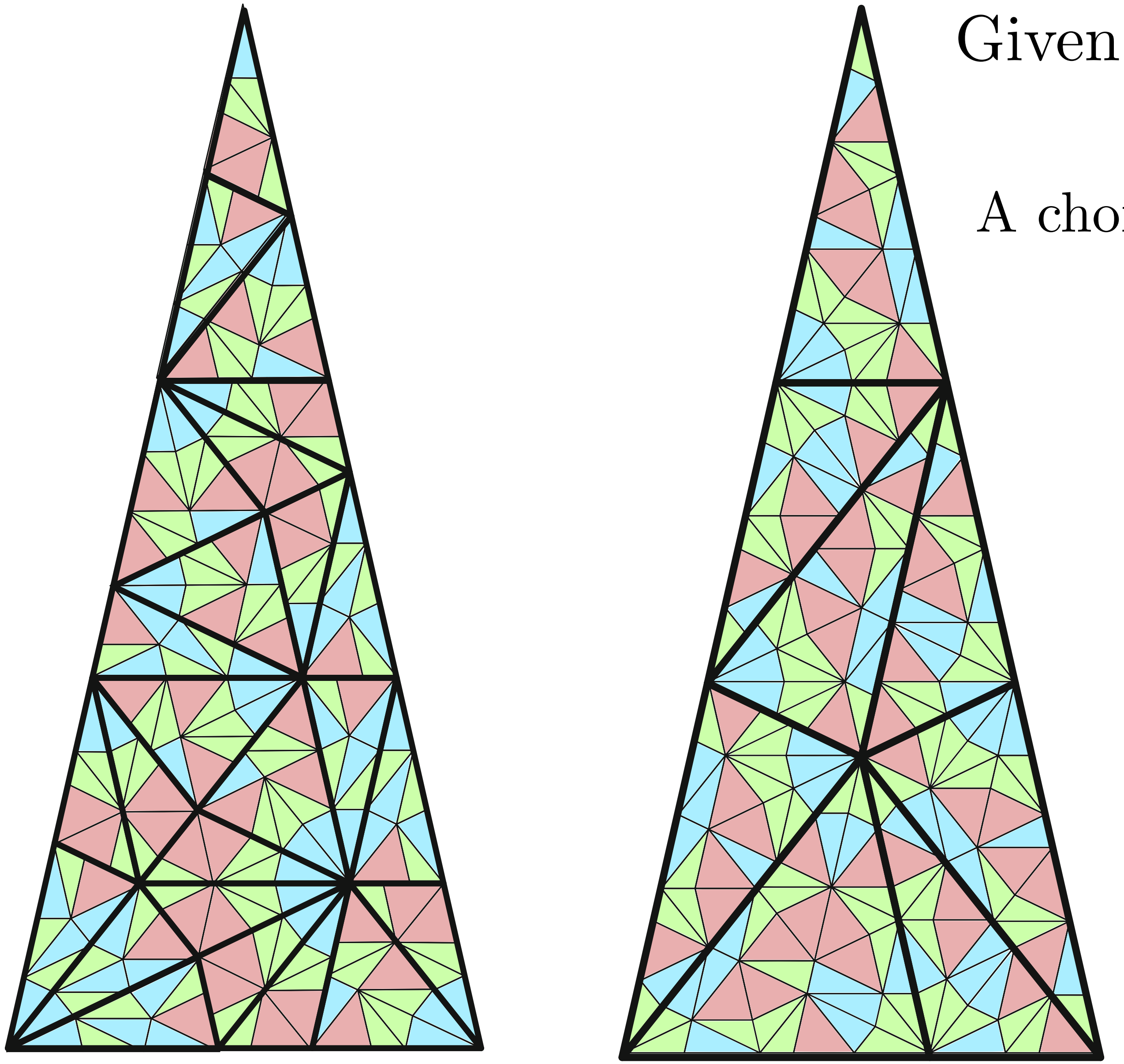

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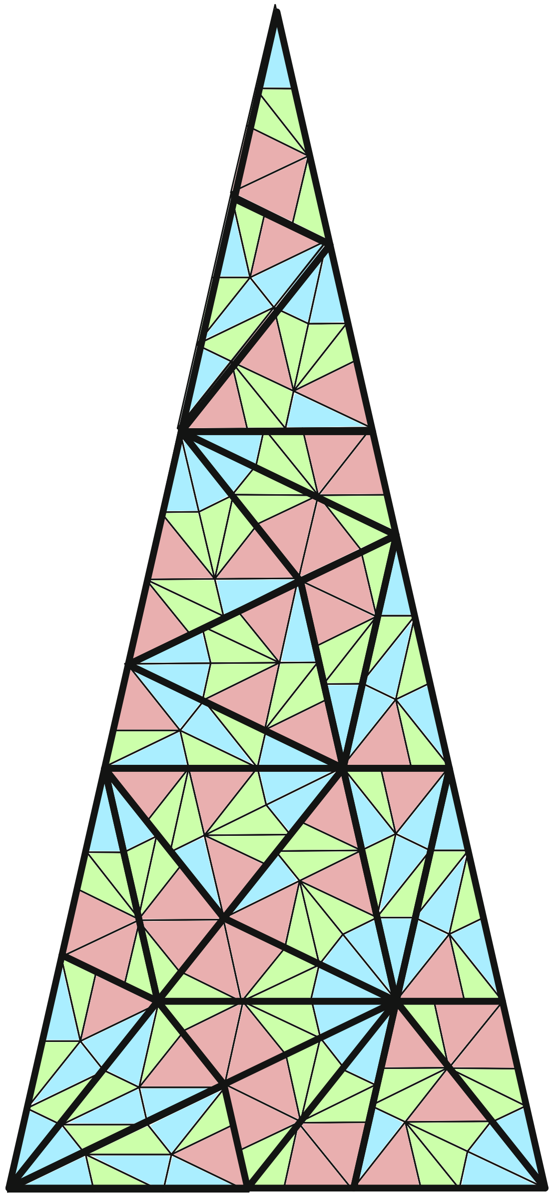

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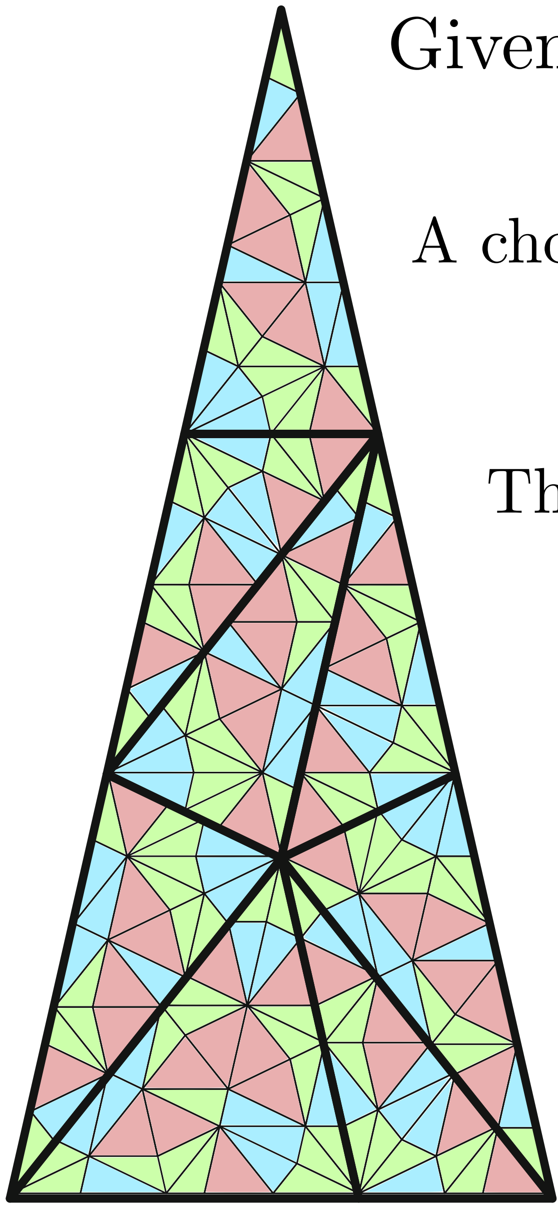

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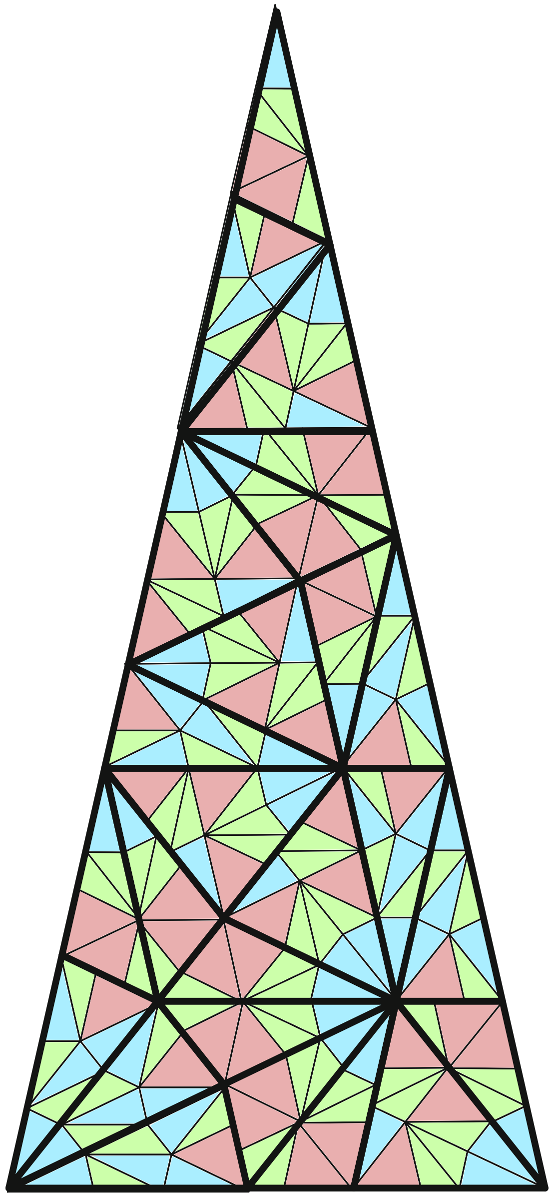

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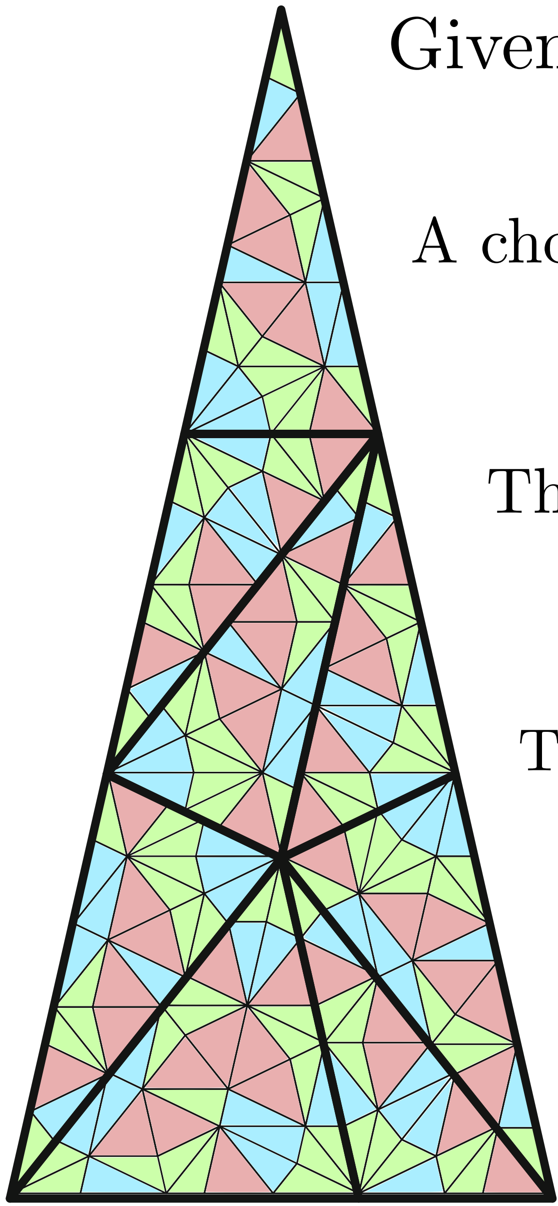

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The shift $\sigma : \Sigma_N \to \Sigma_N$ induces a homeomorphism of tiling spaces $\Phi_x : \Omega_x \to \Omega_{\sigma(x)}$

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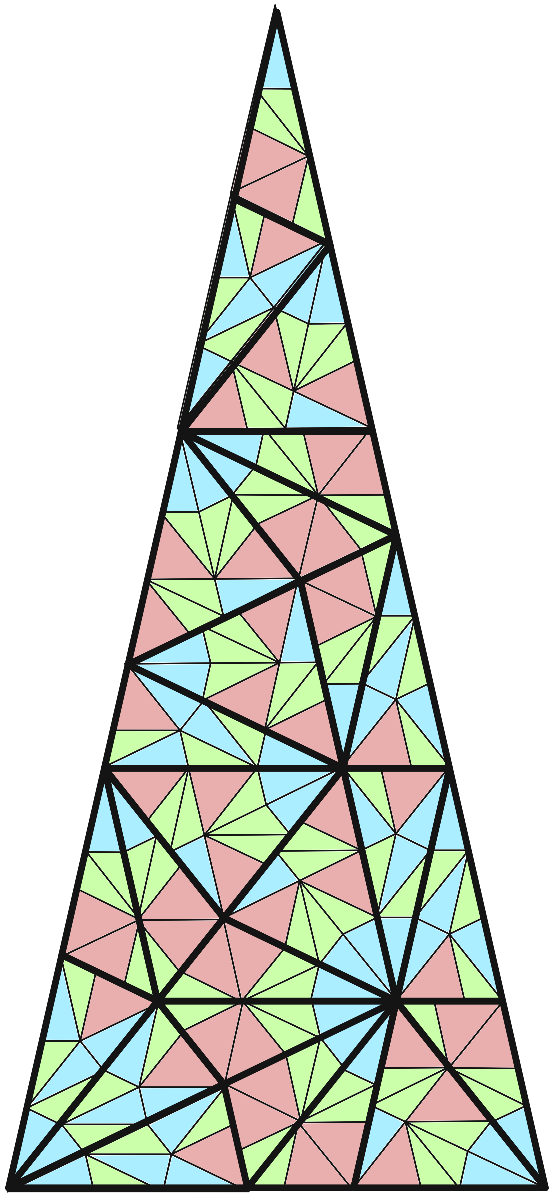

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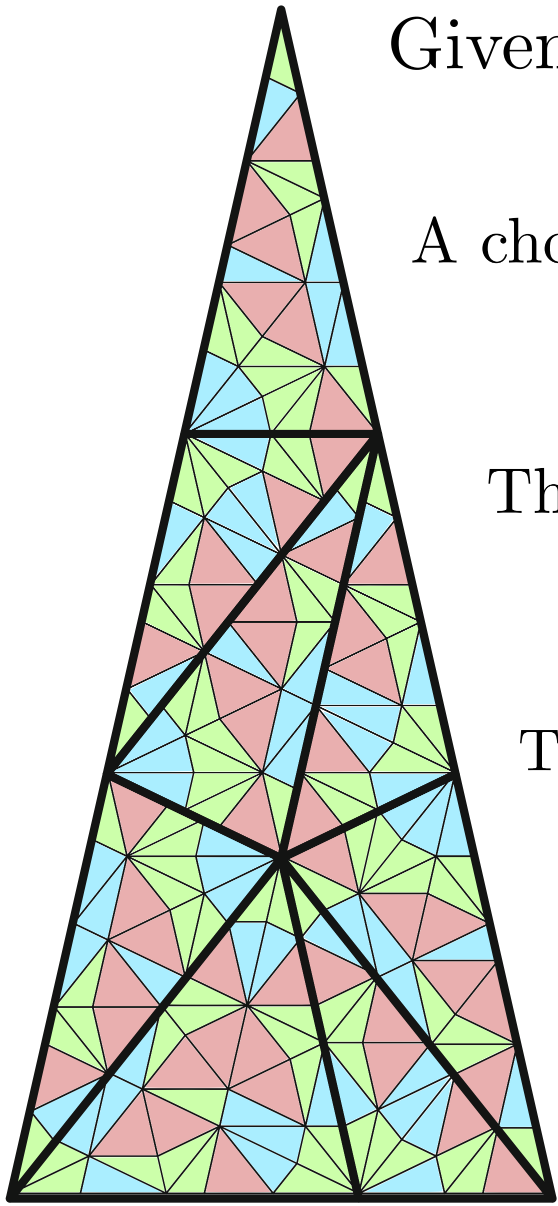

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The shift $\sigma : \Sigma_N \to \Sigma_N$ induces a homeomorphism of tiling spaces $\Phi_x : \Omega_x \to \Omega_{\sigma(x)}$

and induces an action on cohomology

$$\Phi_x^* : H^*(\Omega_{\sigma(x)}; \mathbb{R}) \to H^*(\Omega_x; \mathbb{R})$$

Deformations

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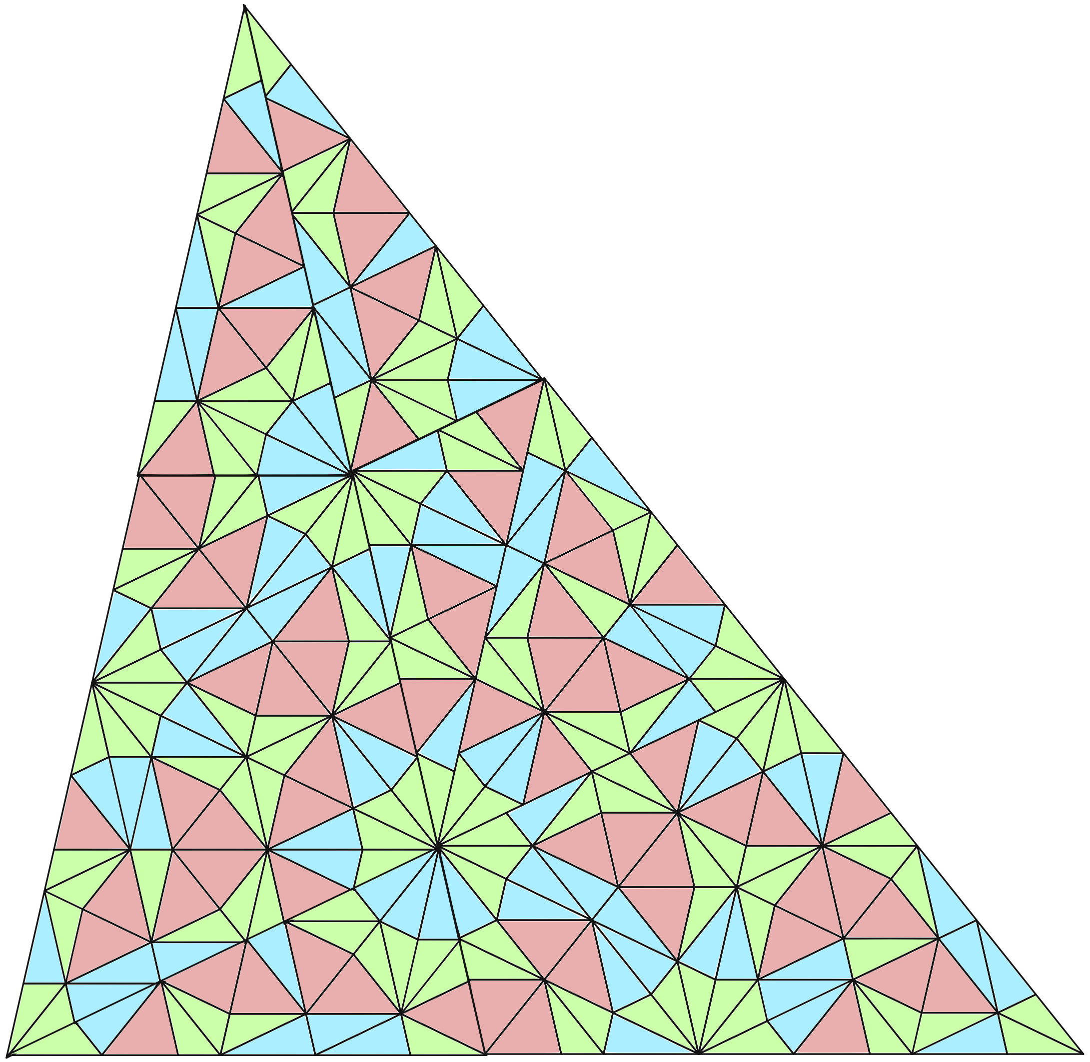

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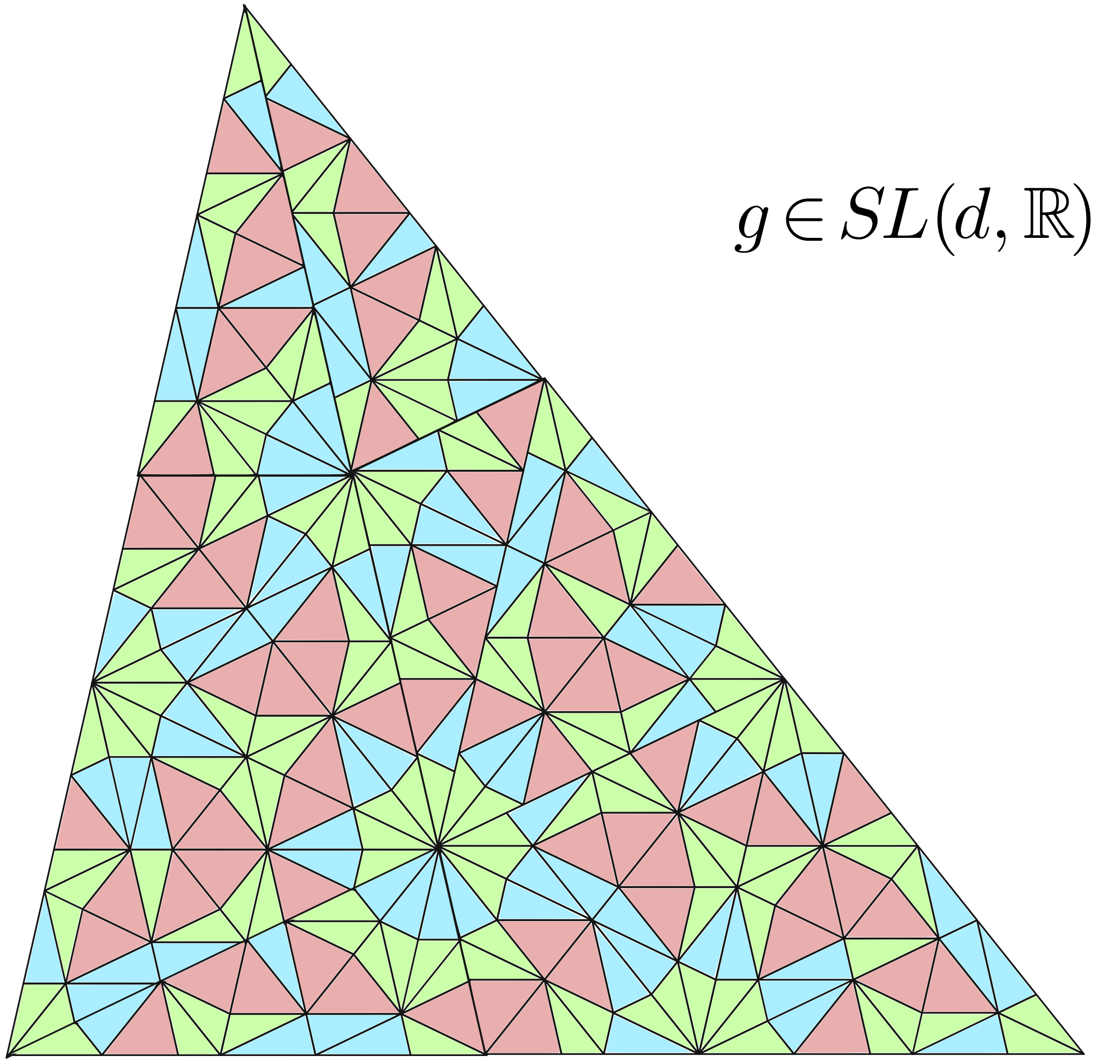

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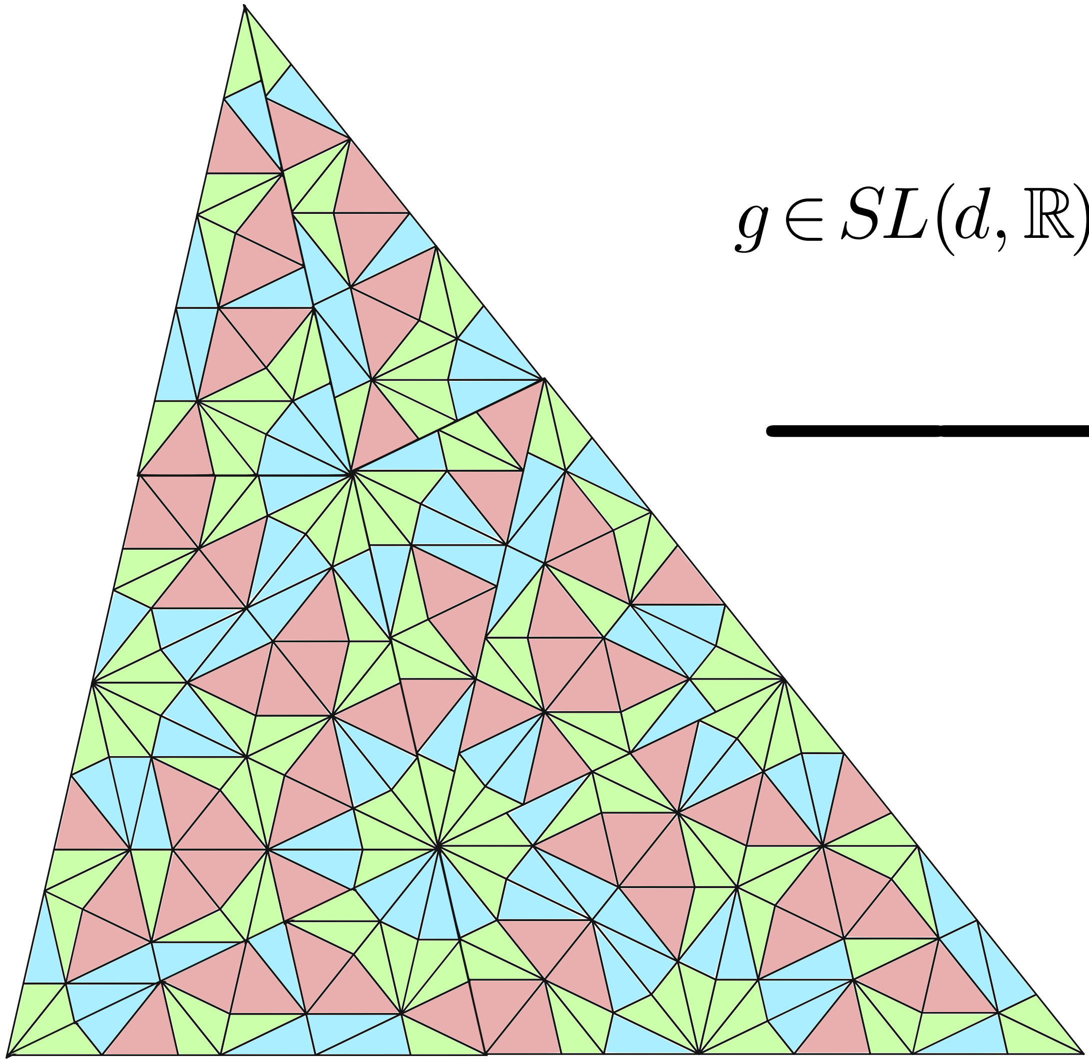

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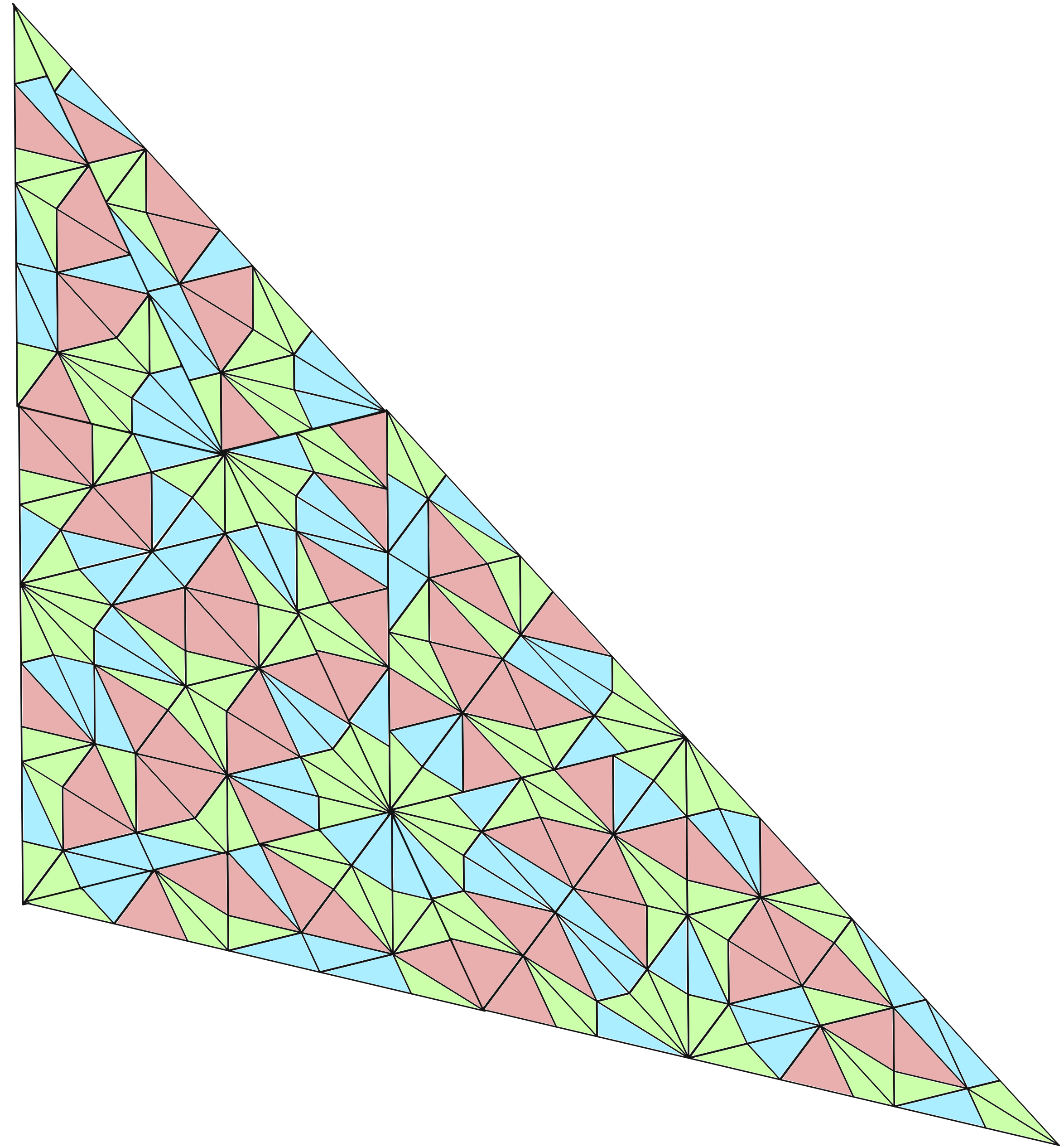

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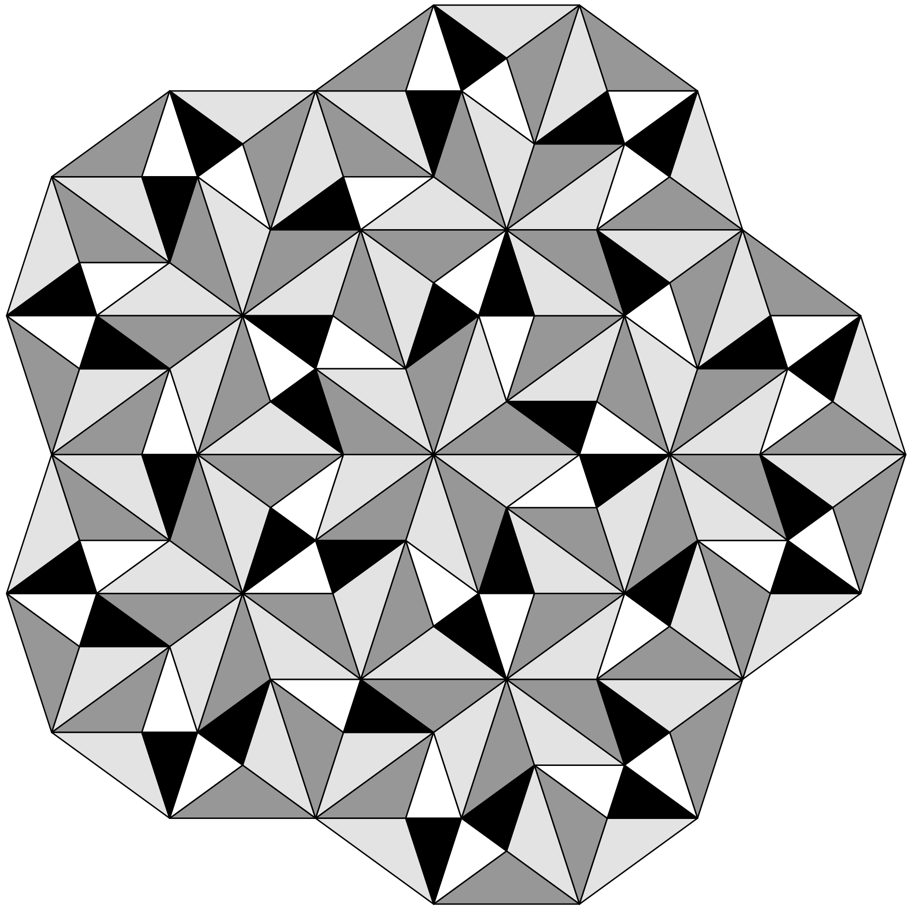

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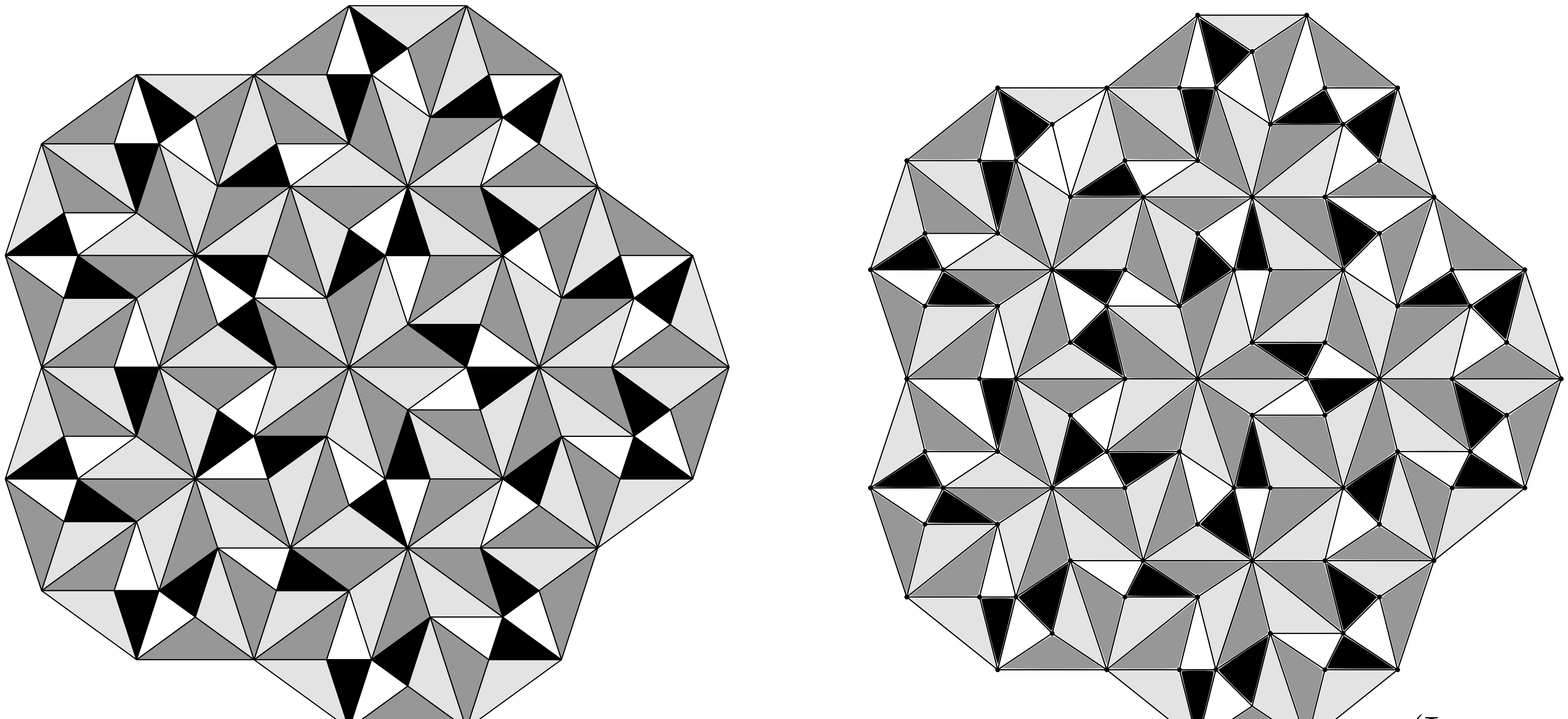

(Image taken from Clark-Sadun 2006)

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There is an open subset $\mathcal{M}_x \subset H^1(\Omega_x;\mathbb{R}^d)$ of non-singular deformations (Julien-Sadun)

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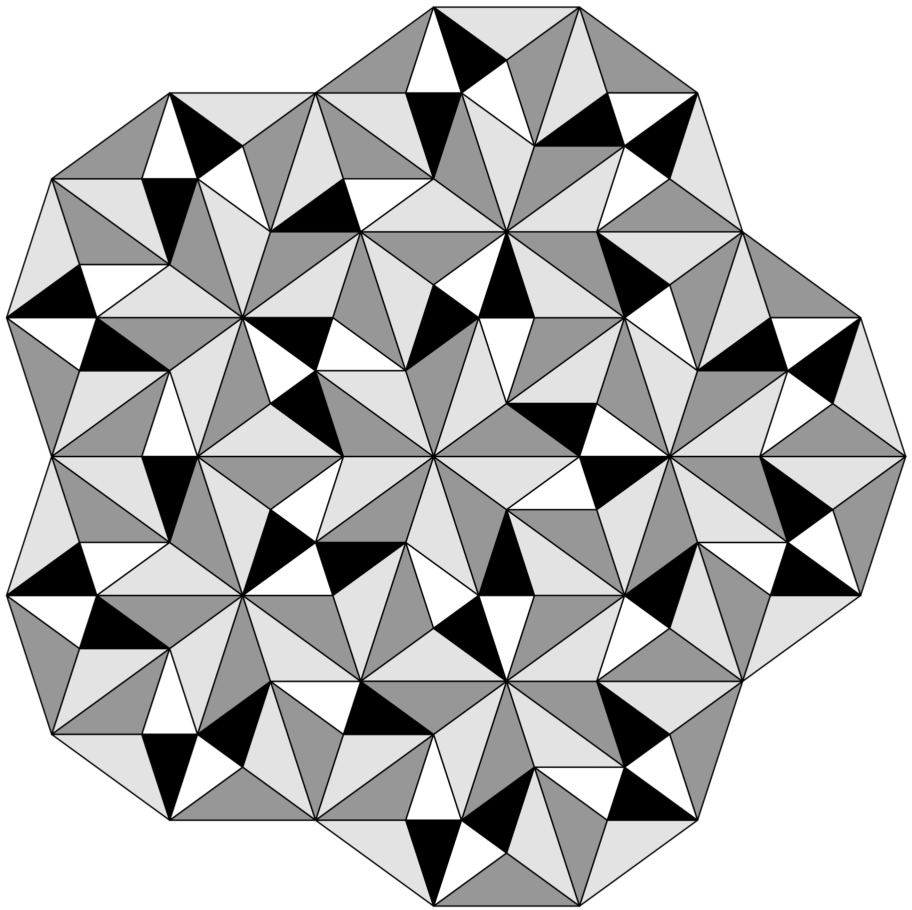

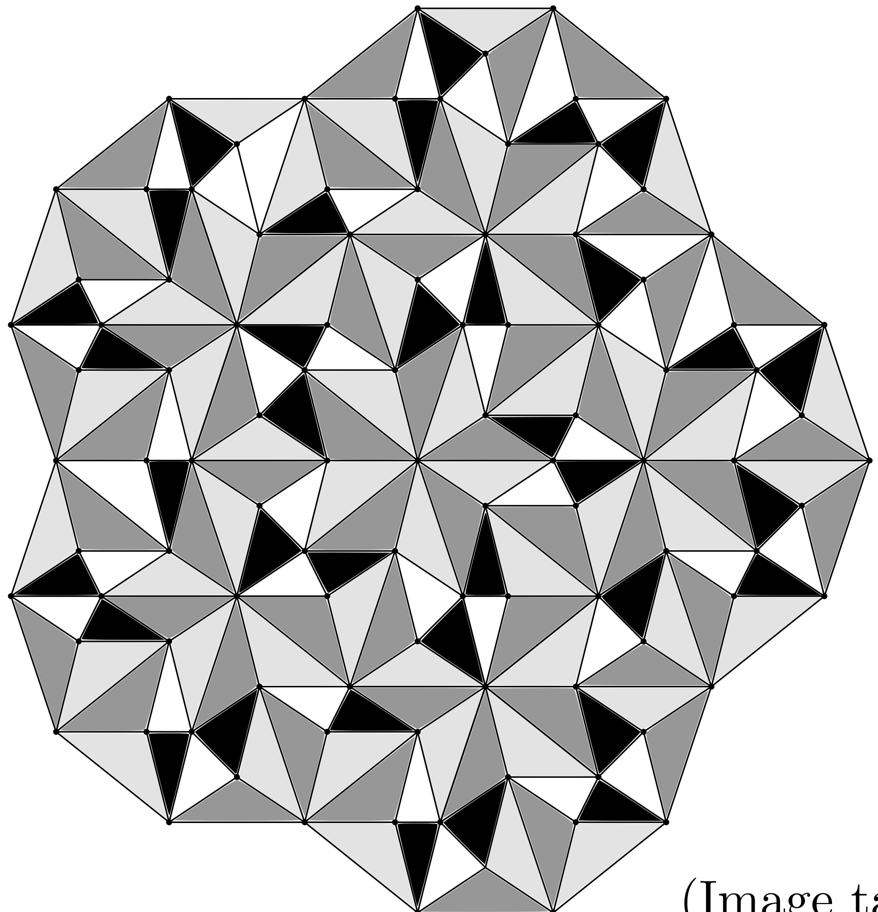

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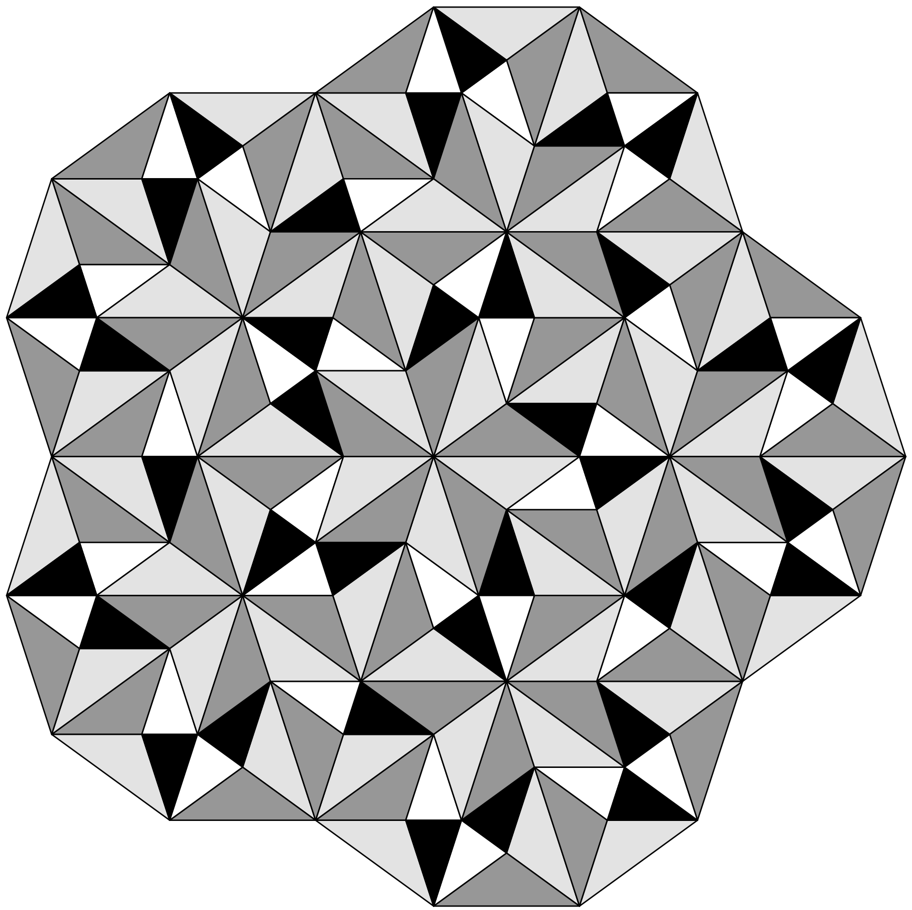

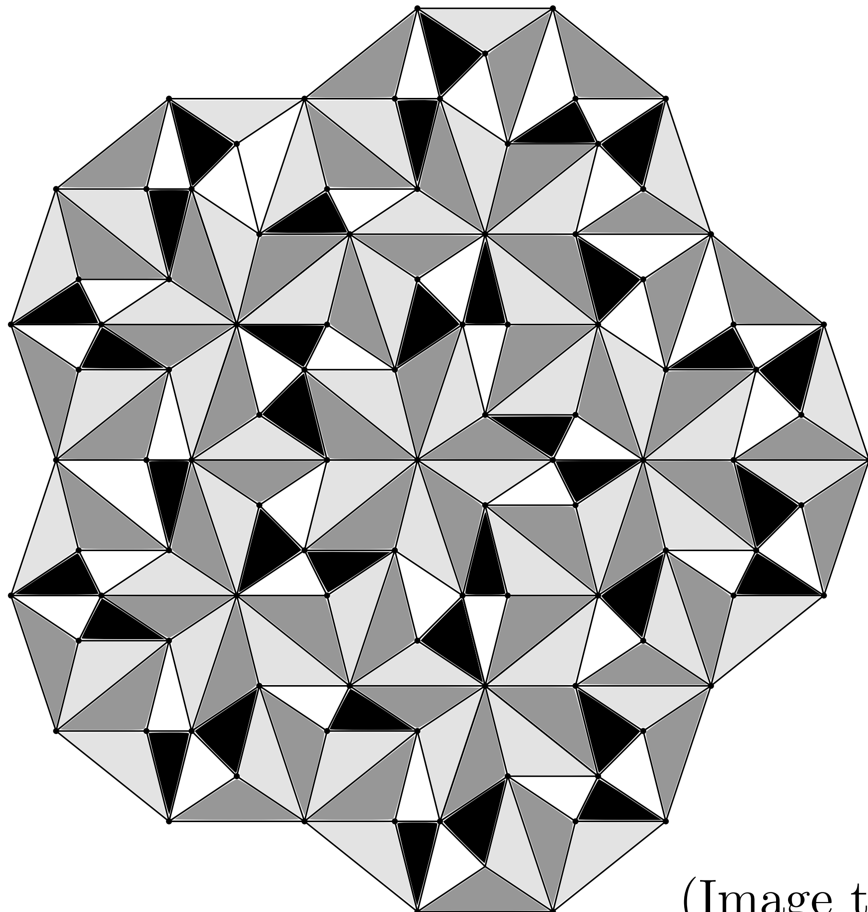

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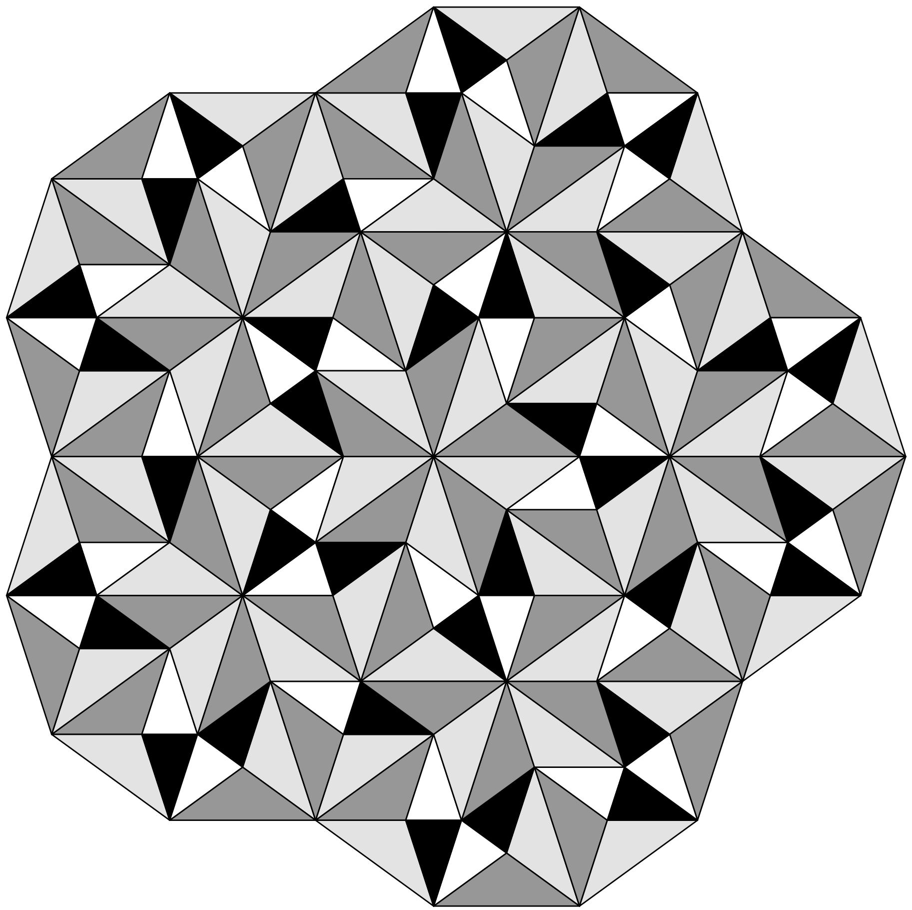

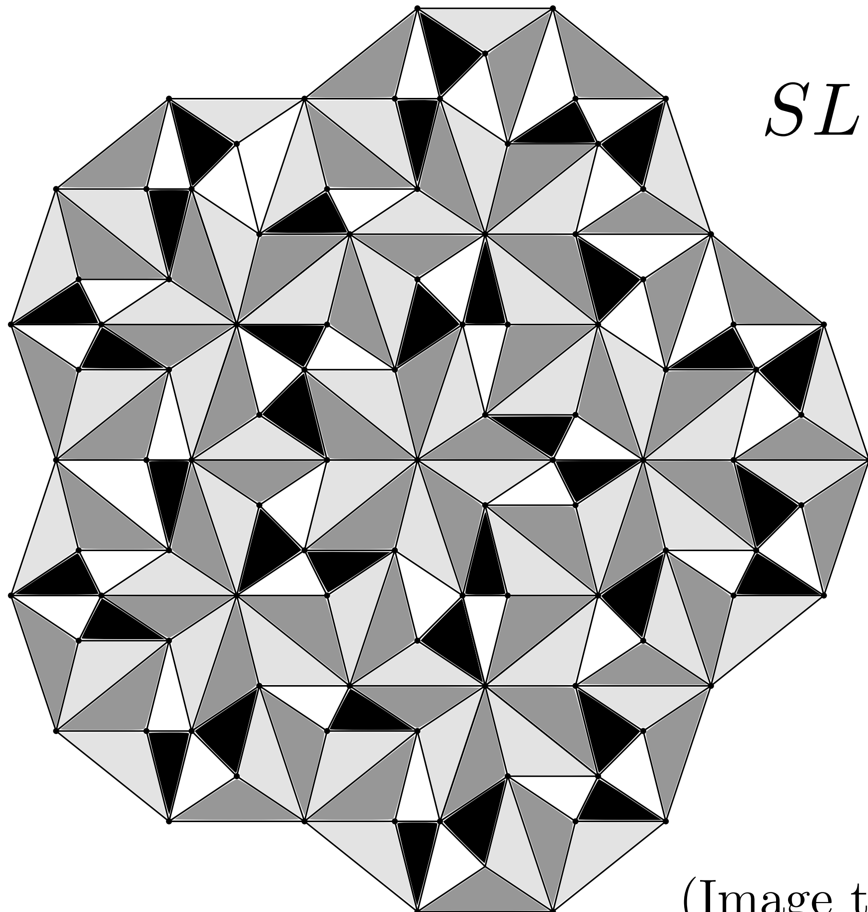

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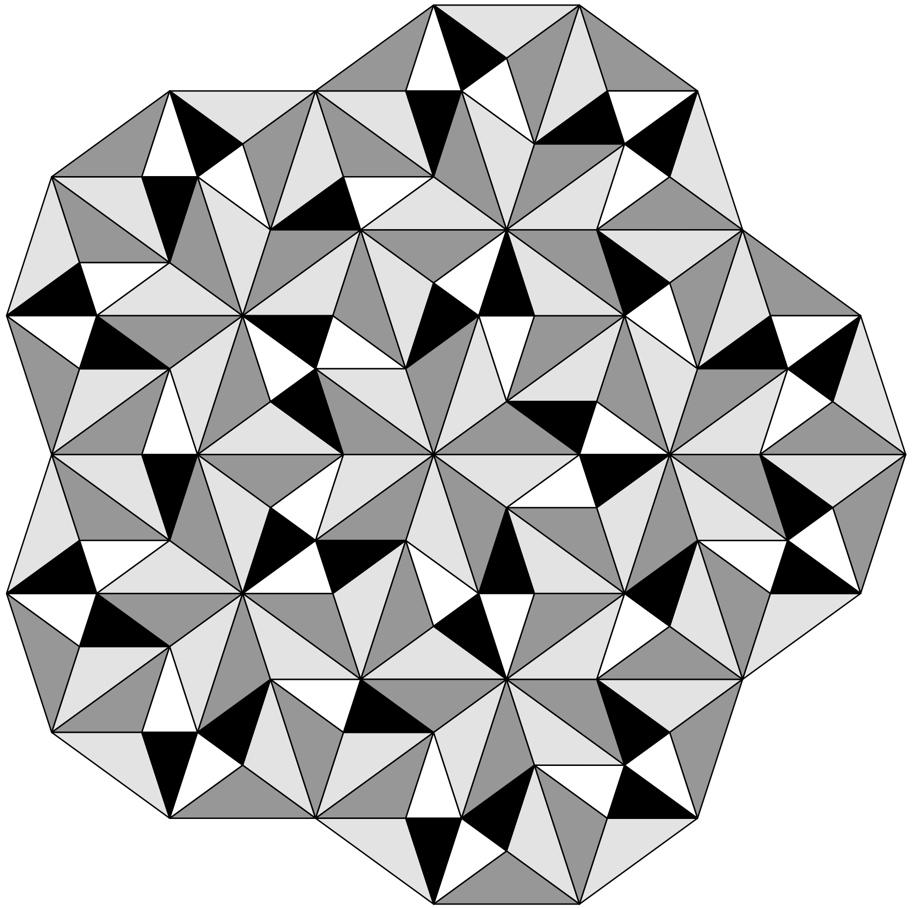

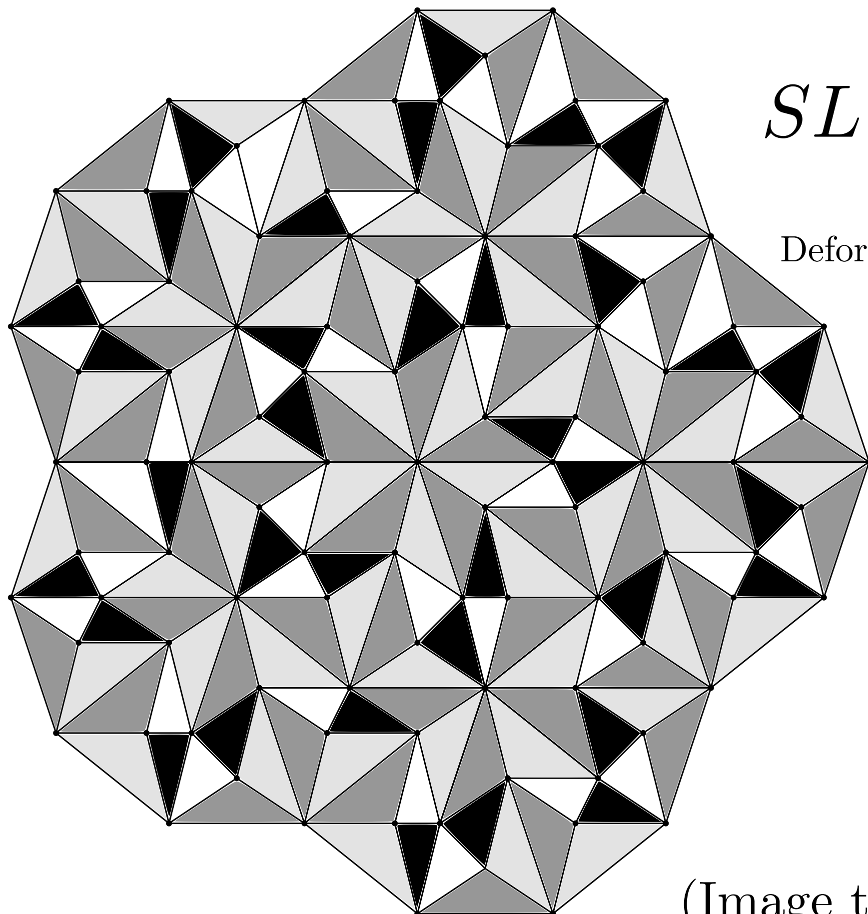

$SL(d,\mathbb{R}) \subset H^1(\Omega_x;\mathbb{R}^d)$

Deformations give deformations of tiling spaces, changing the $\mathbb{R}^d$ action and changing the geometry of return vectors.

(Image taken from Clark-Sadun 2006)

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for all R large enough. In particular, the lower local dimension of the spectral measures is bounded from below:

$$2\alpha_\mu \leq d_f^-(\lambda).$$

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and **for all** $\lambda \in \mathbb{R}^d$:

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$\lambda_1 > \lambda_2$ are the top two Lyapunov exponents of the induced cocycle on $H^d(\Omega_x;\mathbb{R})$

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- Estimate Hausdorff dimension of bad deformation parameters and show that it is codimention $\dim(E_x^+) - d$ in space of parameters $\mathcal{M}_x$ (following Bufetov-Solomyak)

Substitution rules

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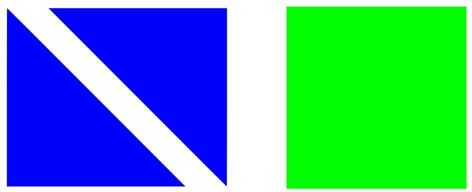

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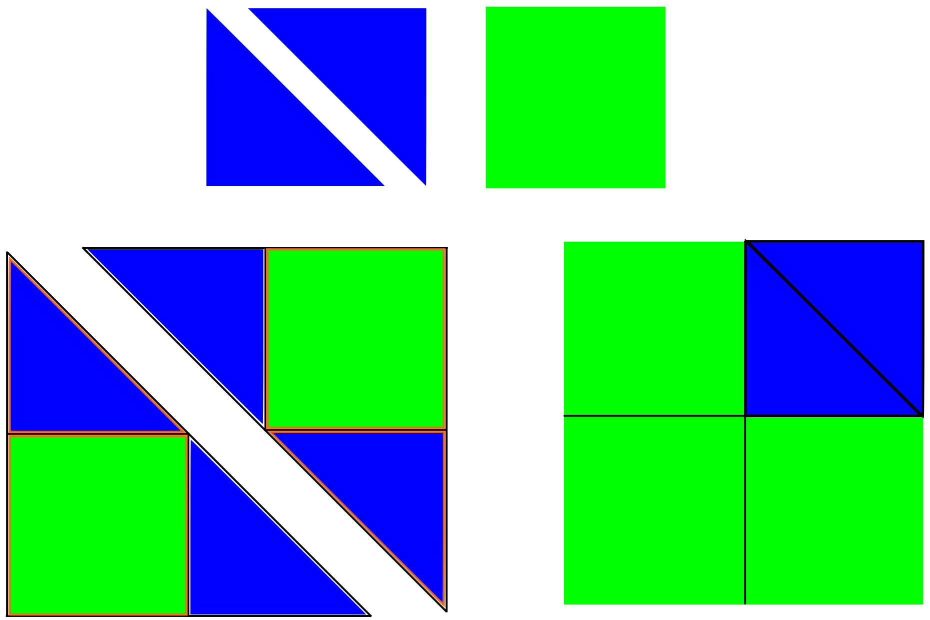

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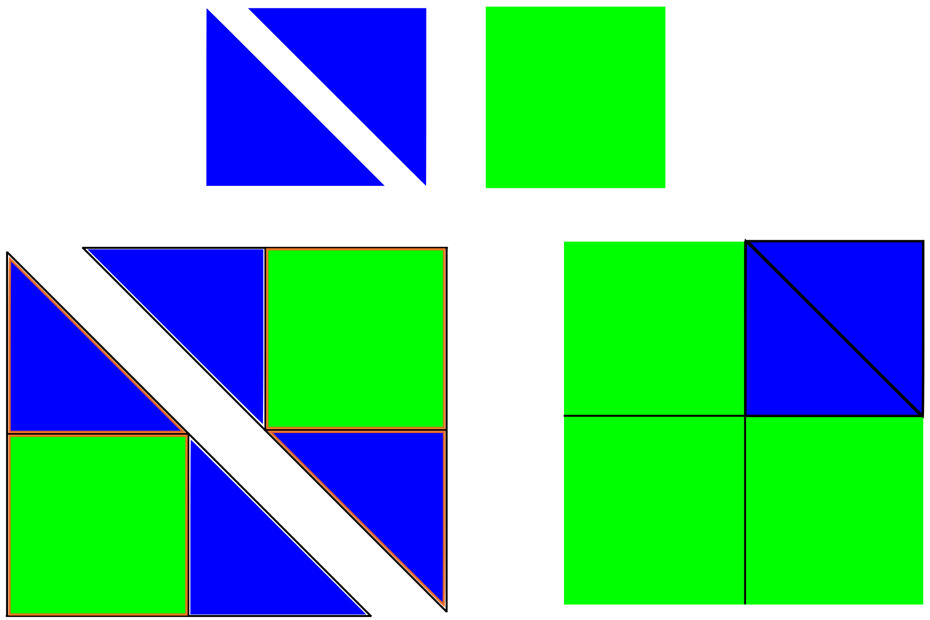

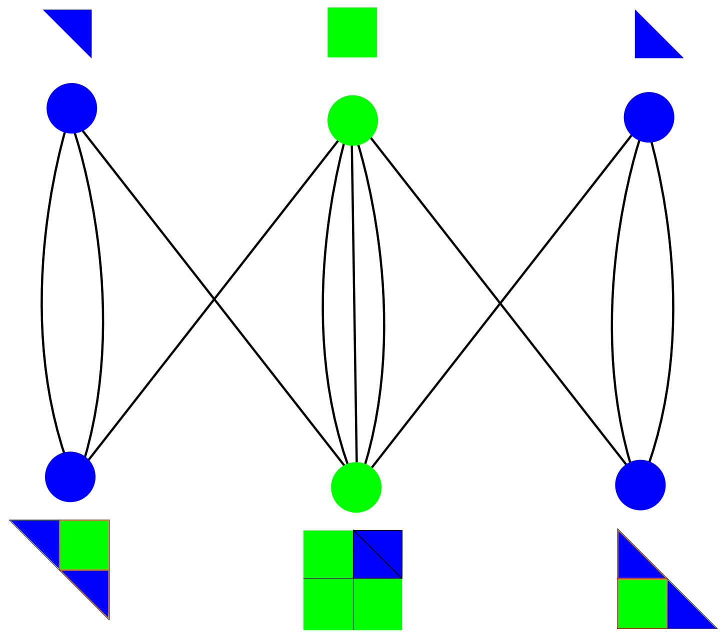

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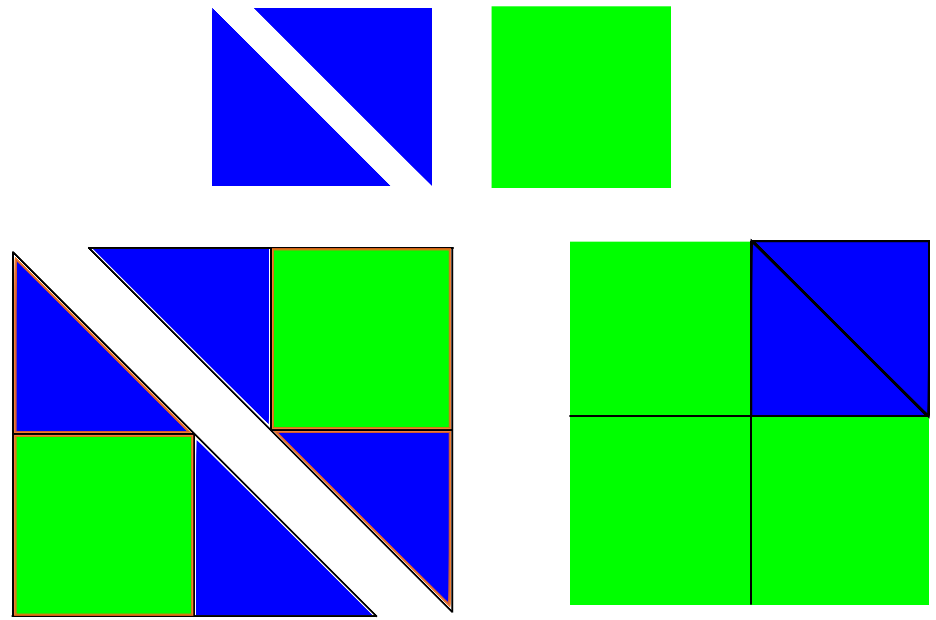
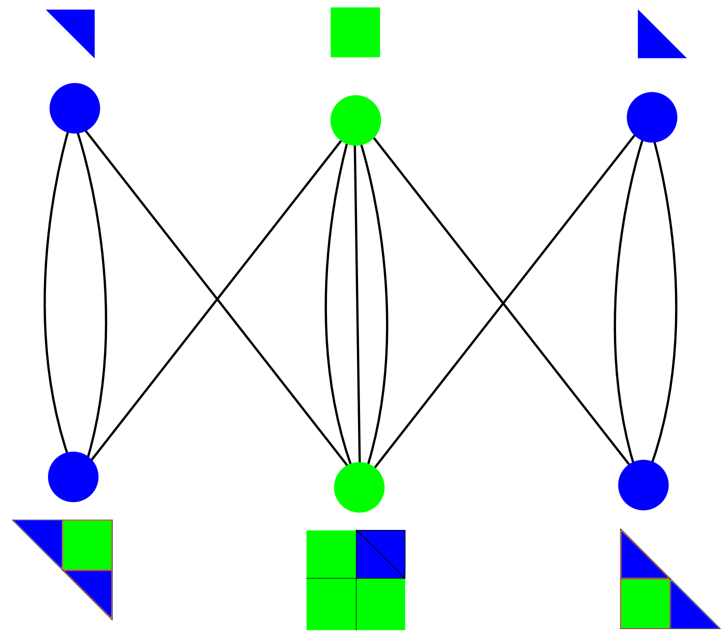
Note: The number of incoming edges is the number of tiles in the substitution





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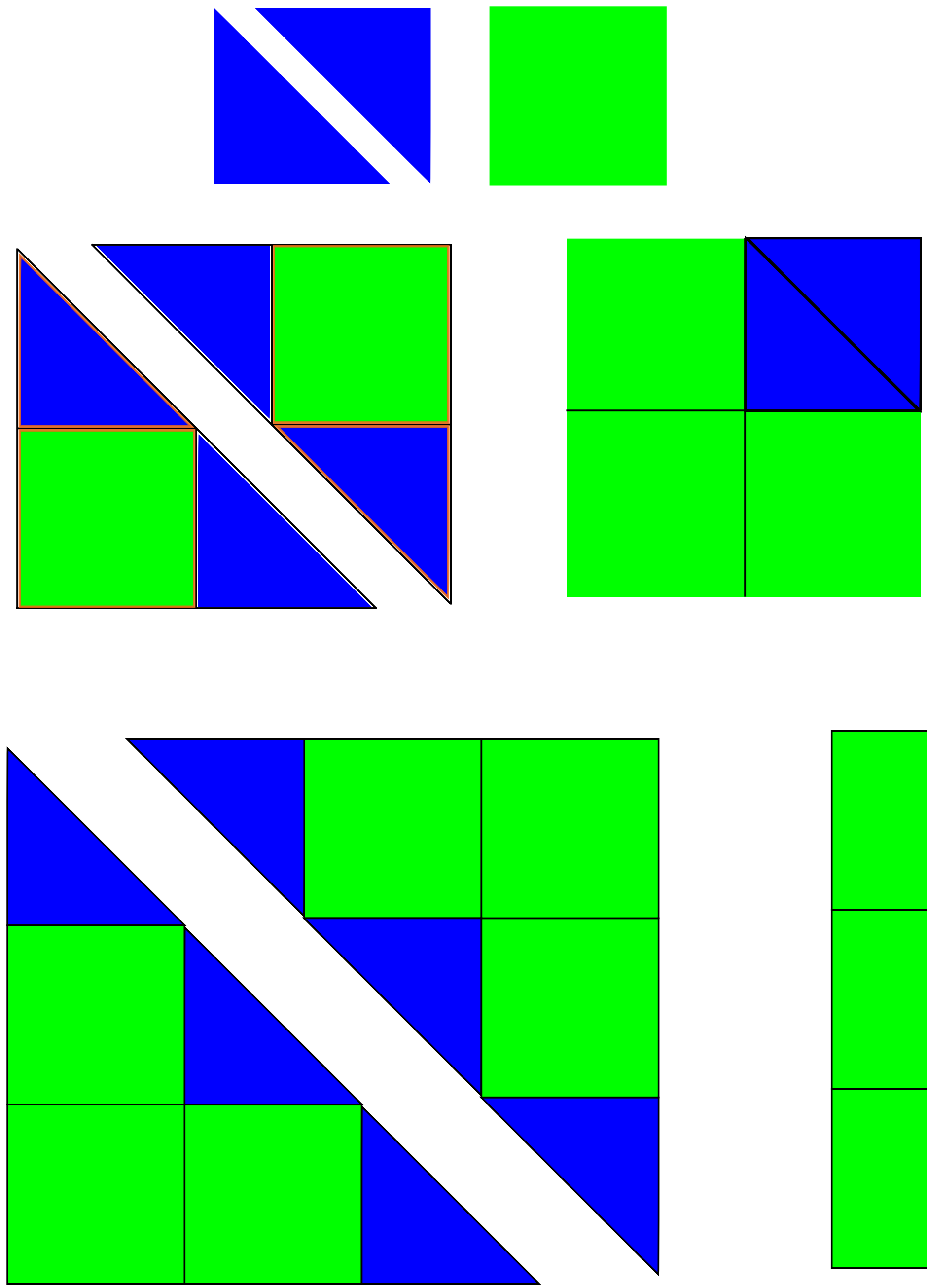

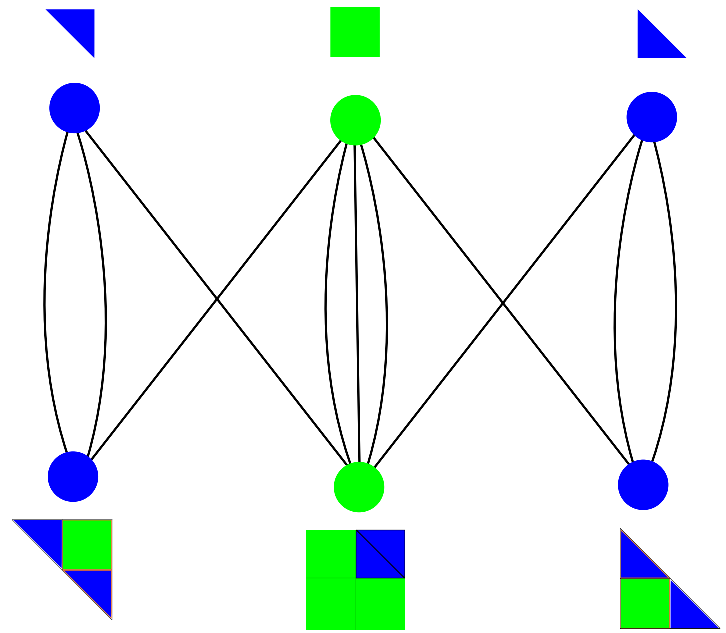

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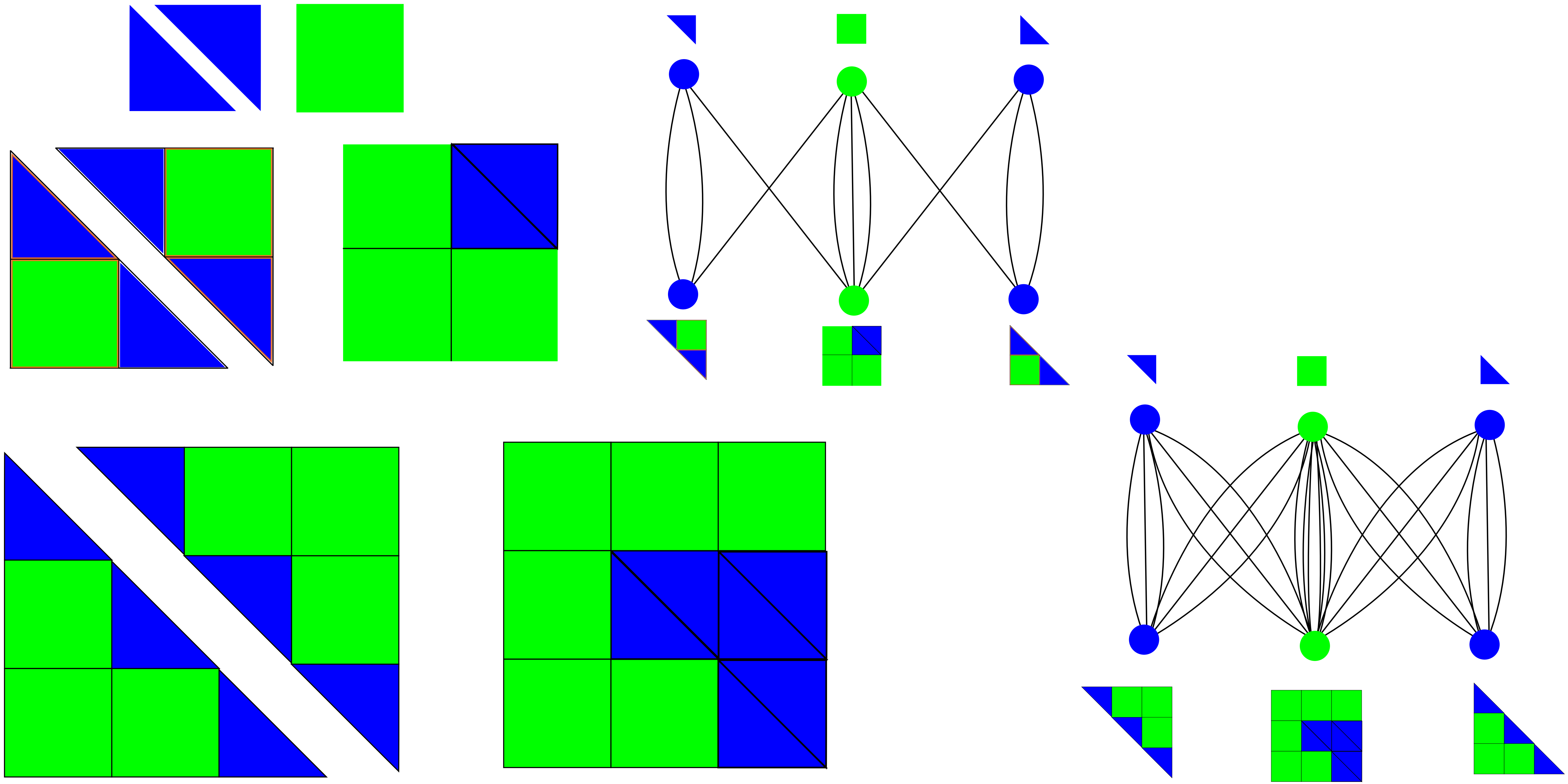

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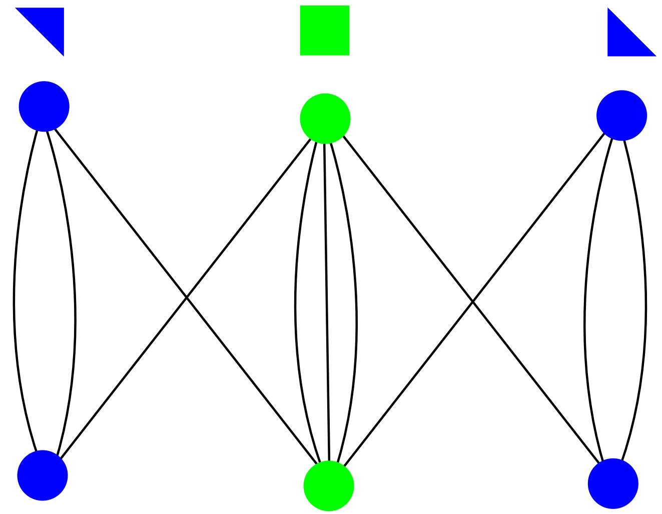

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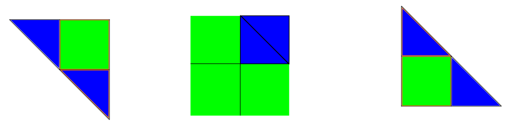

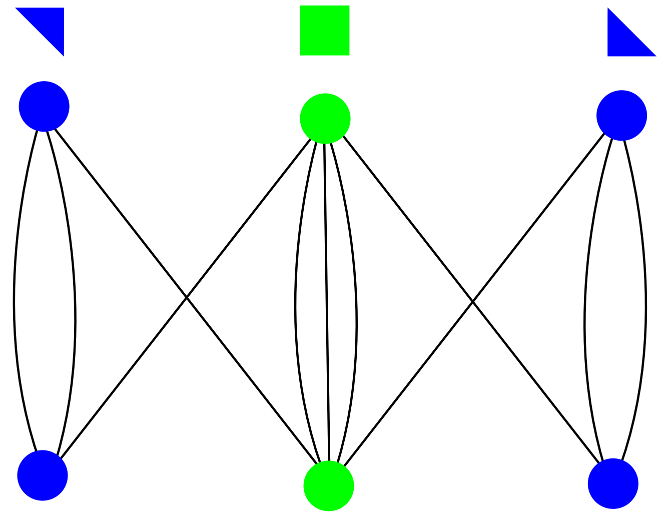

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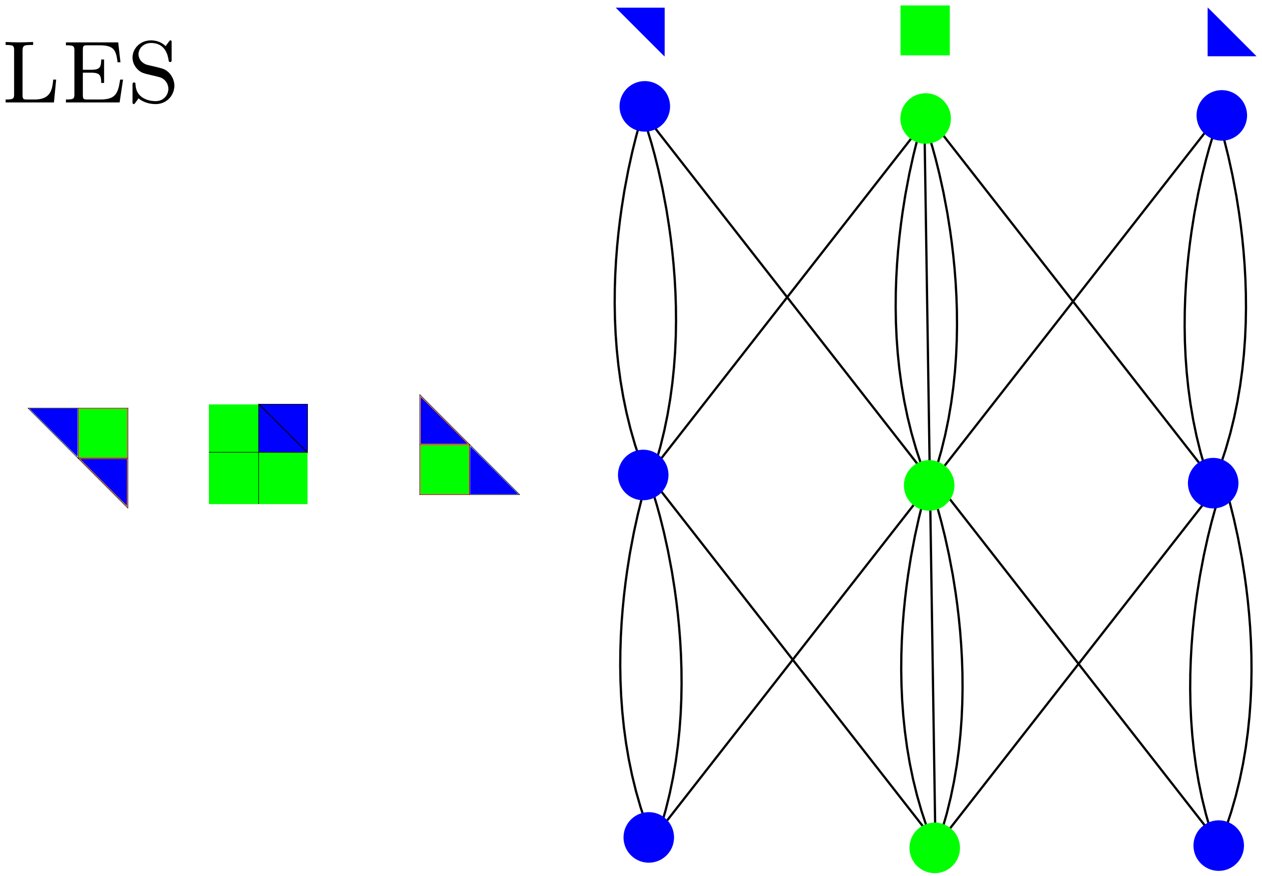

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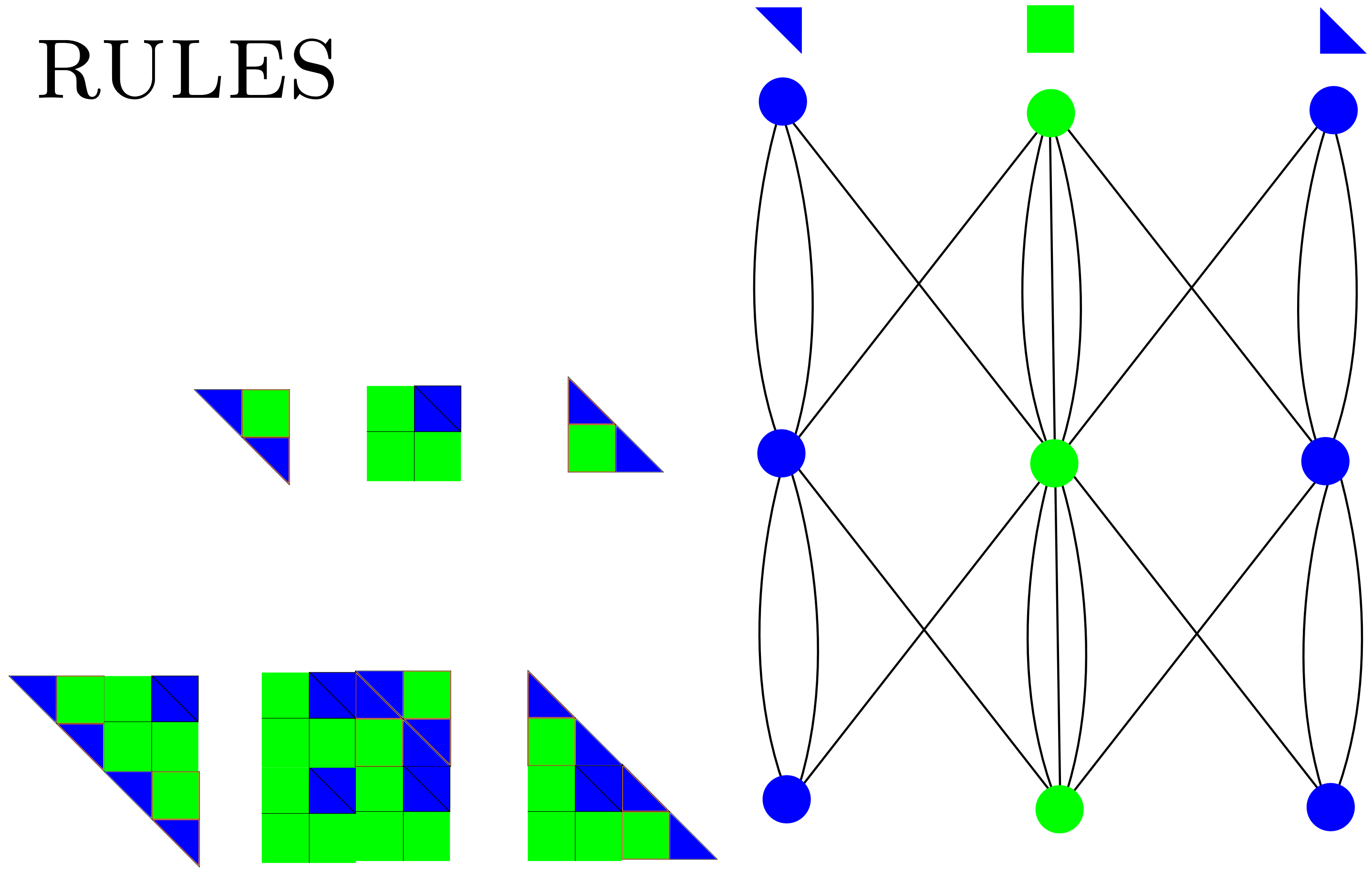

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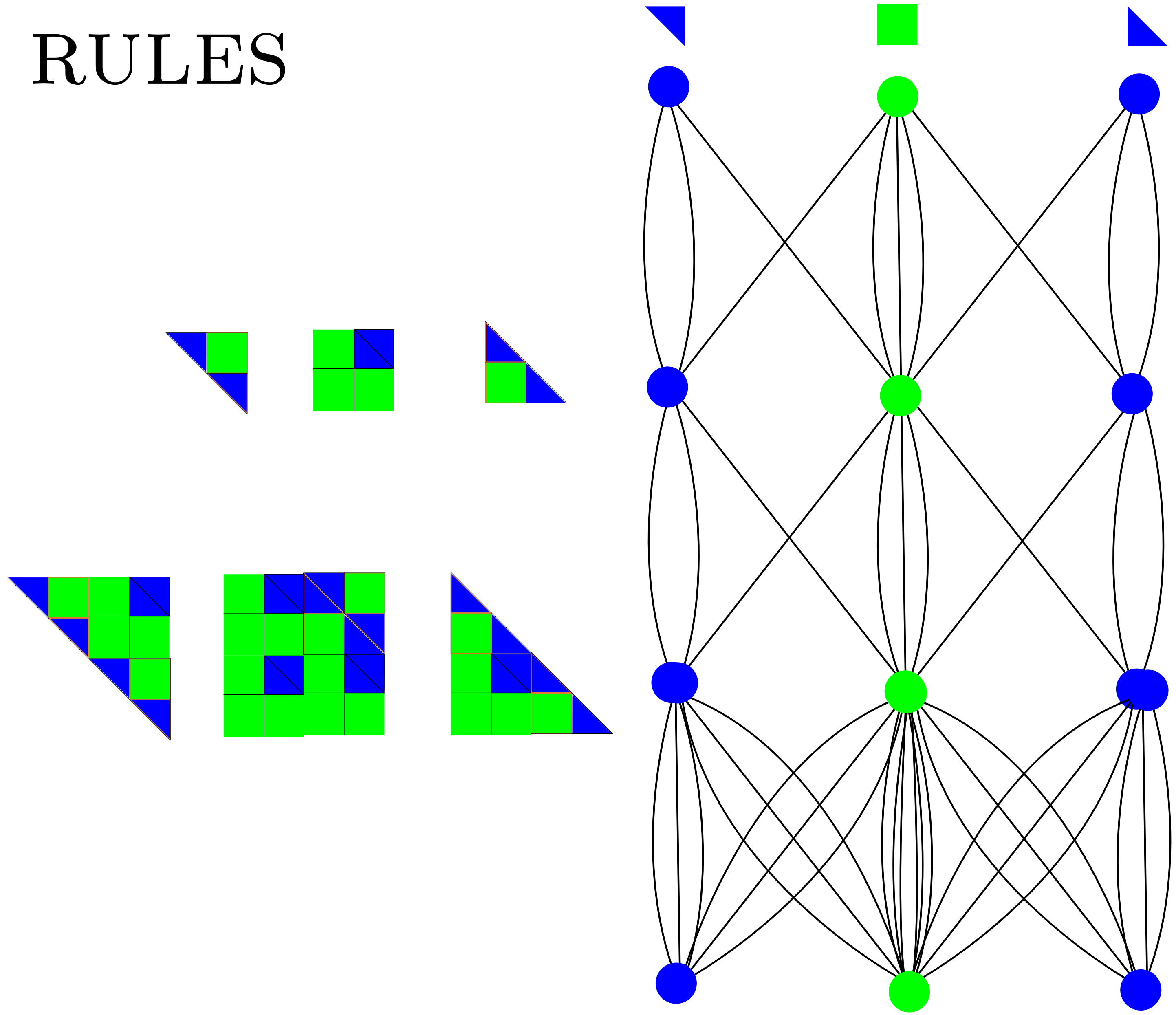

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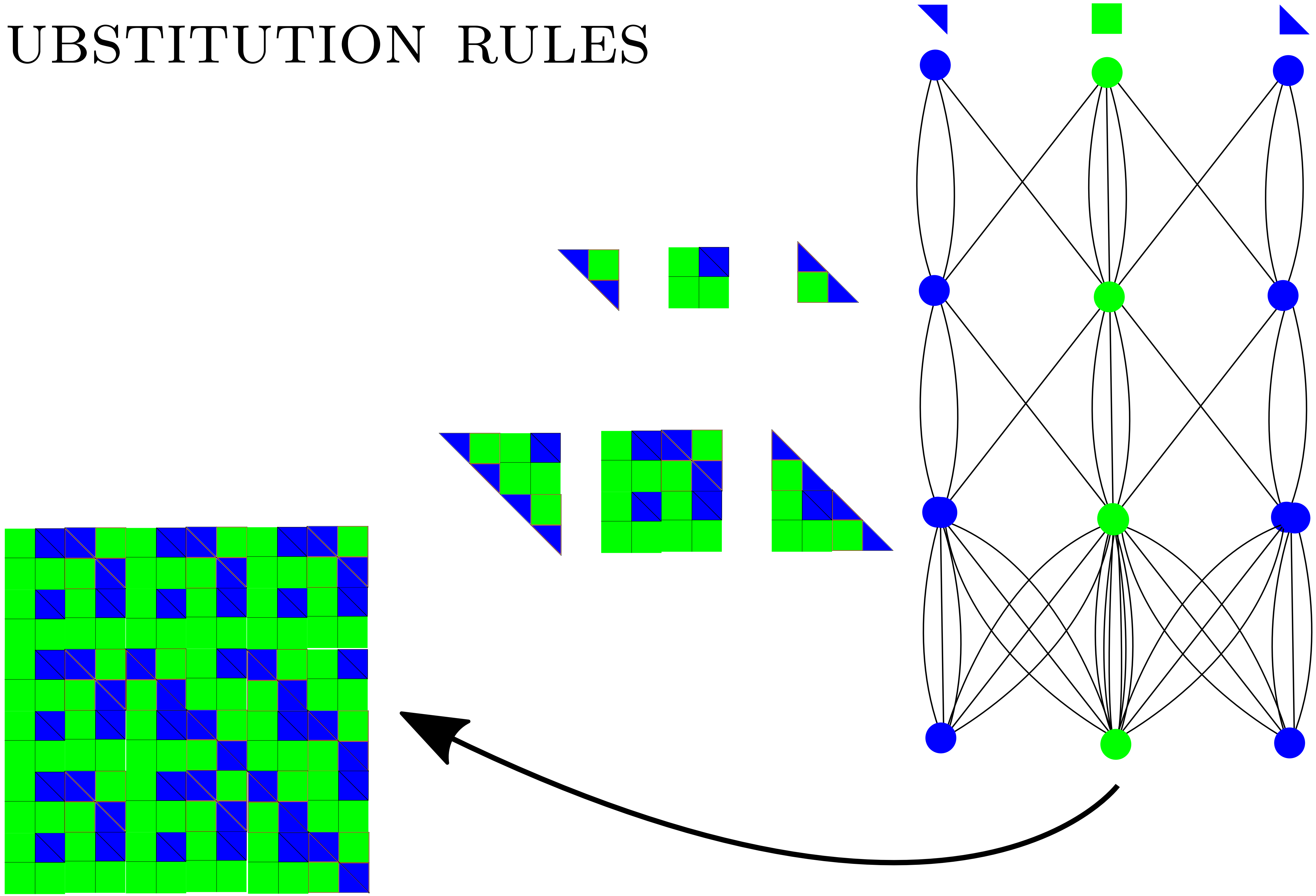

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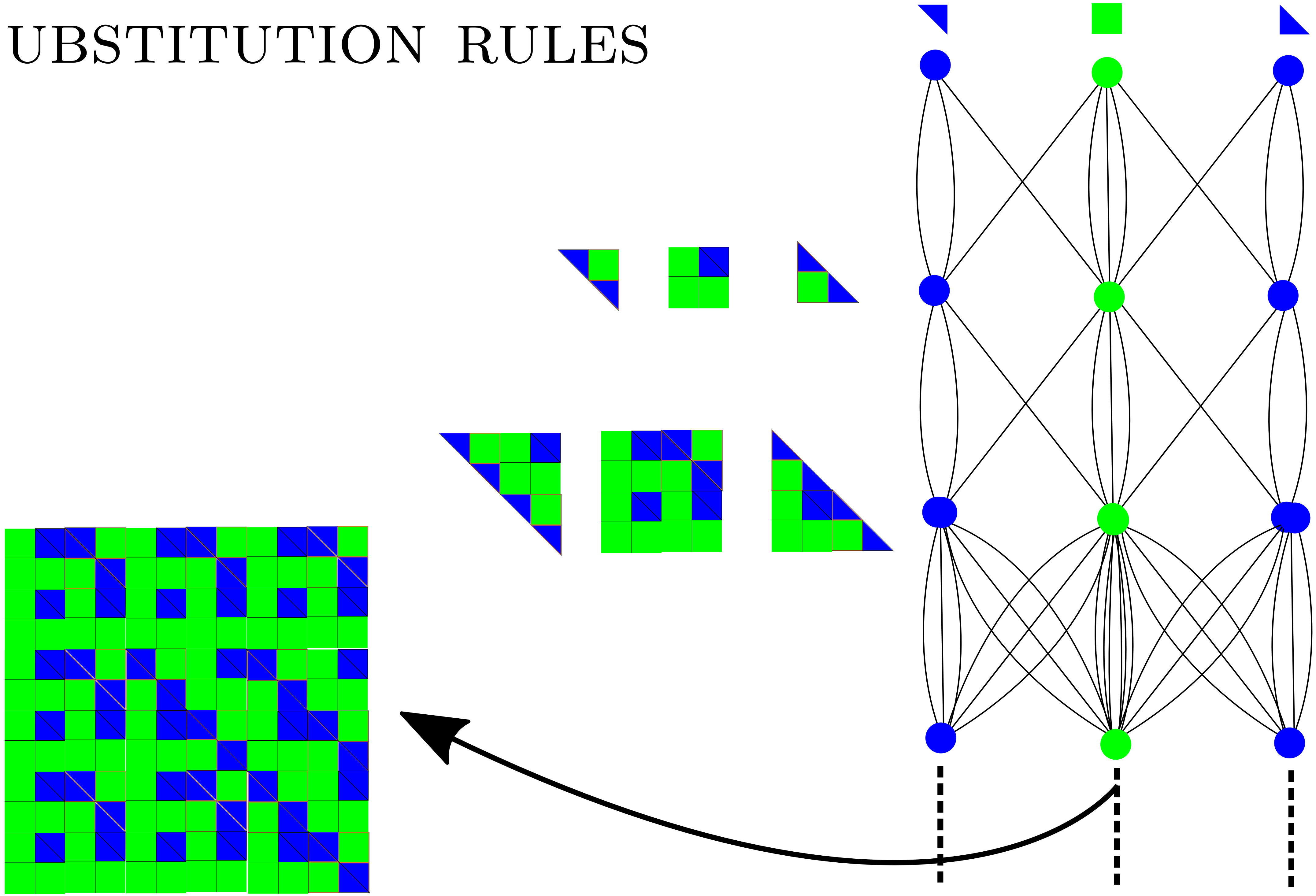

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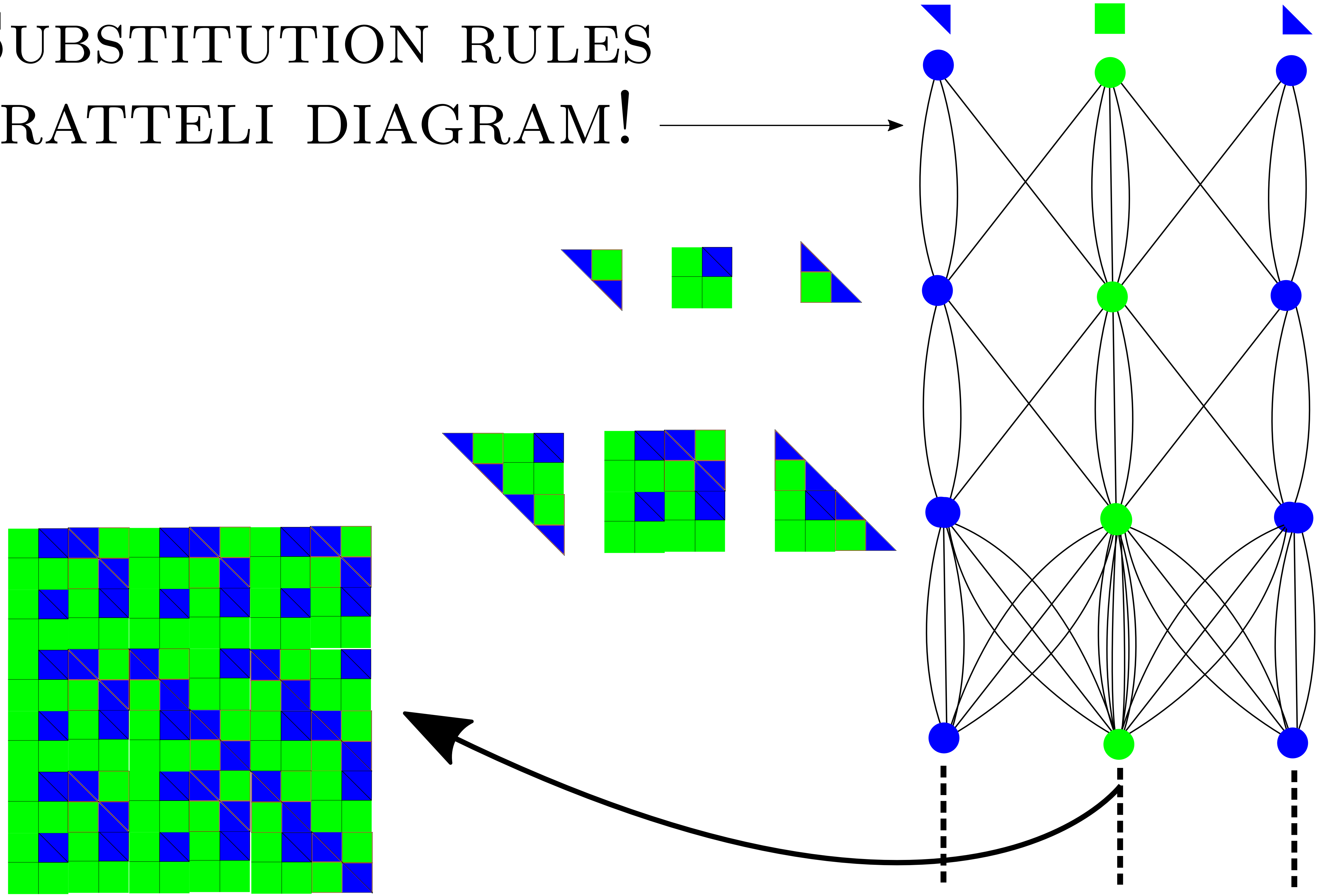
Bratteli diagram!



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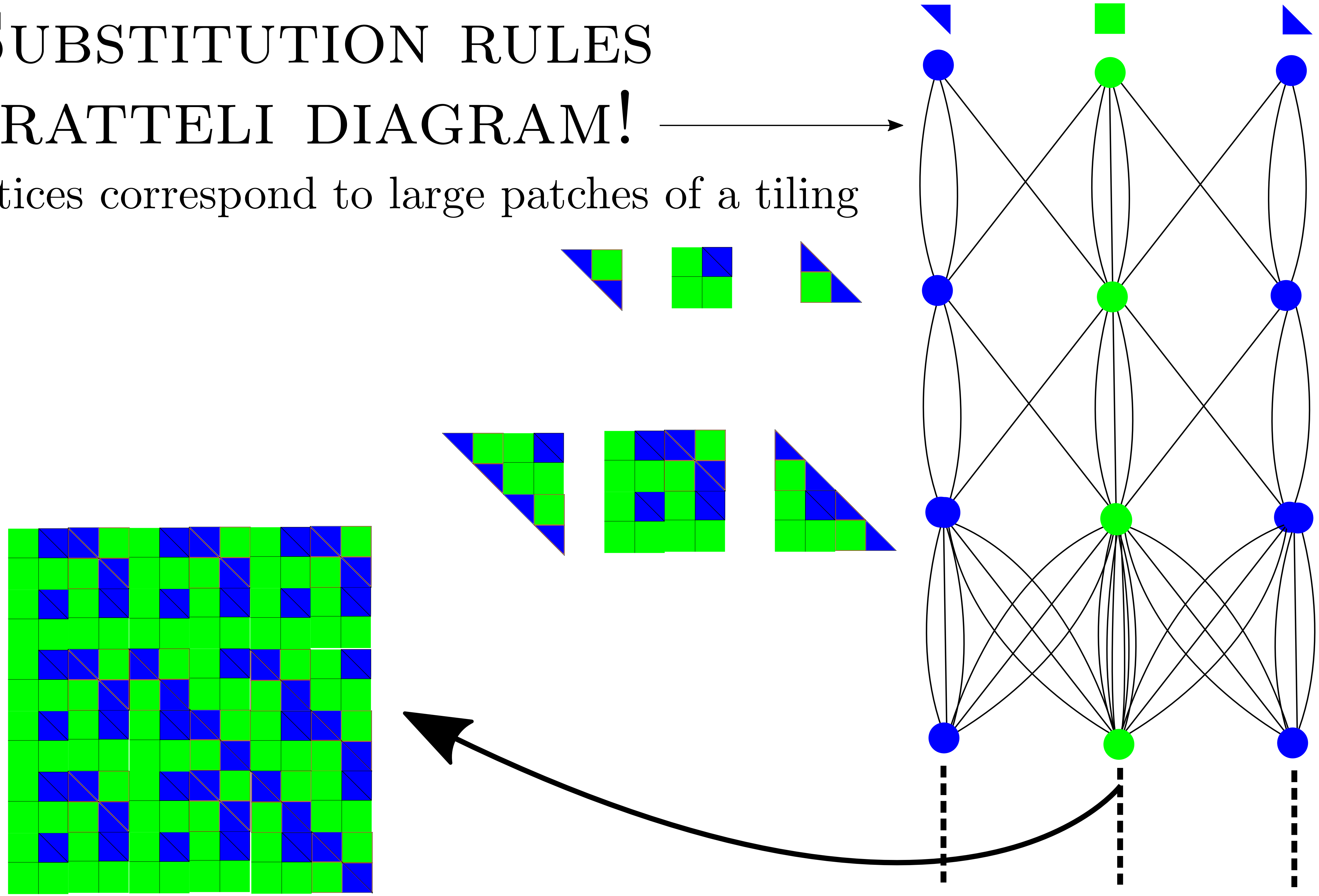
Vertices correspond to large patches of a tiling



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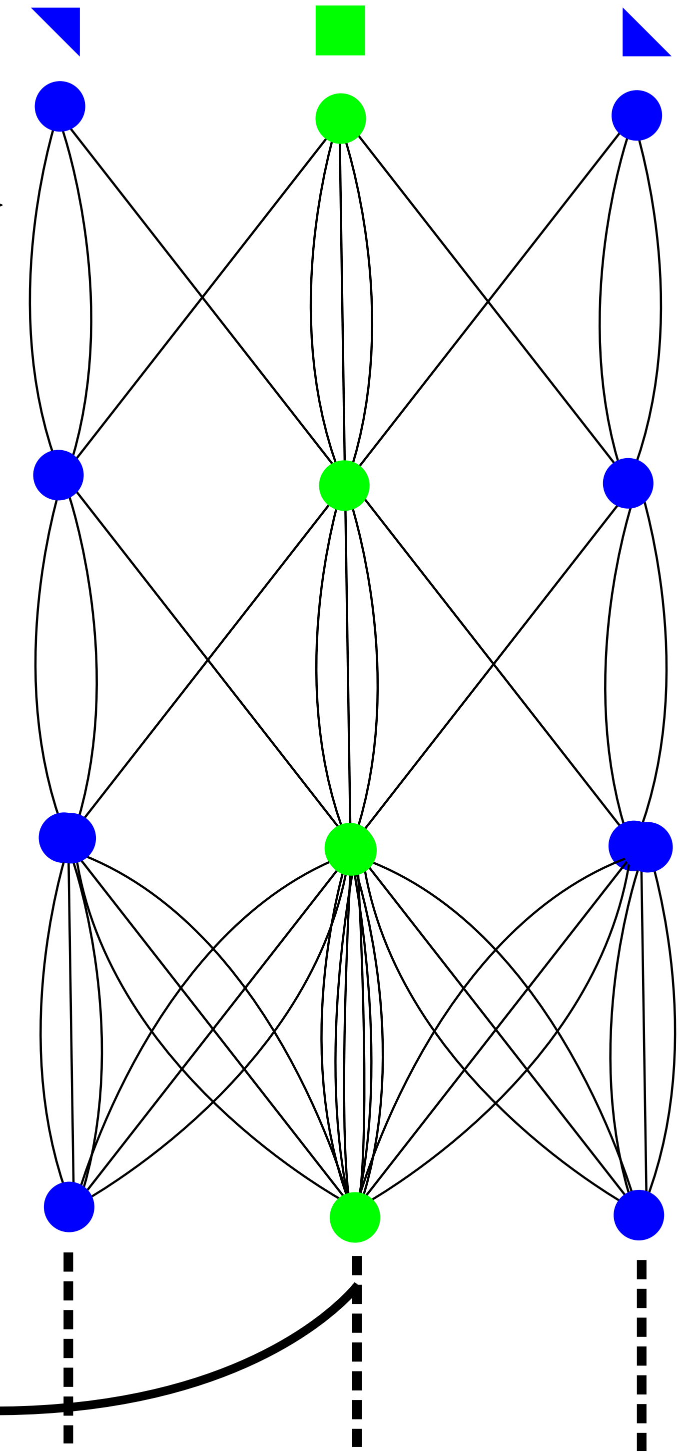

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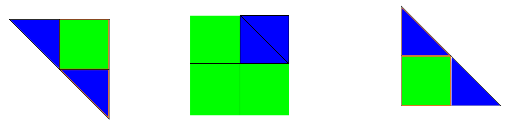
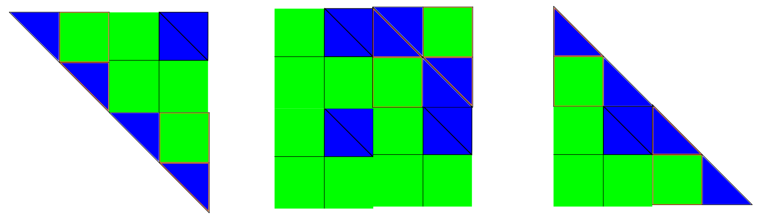
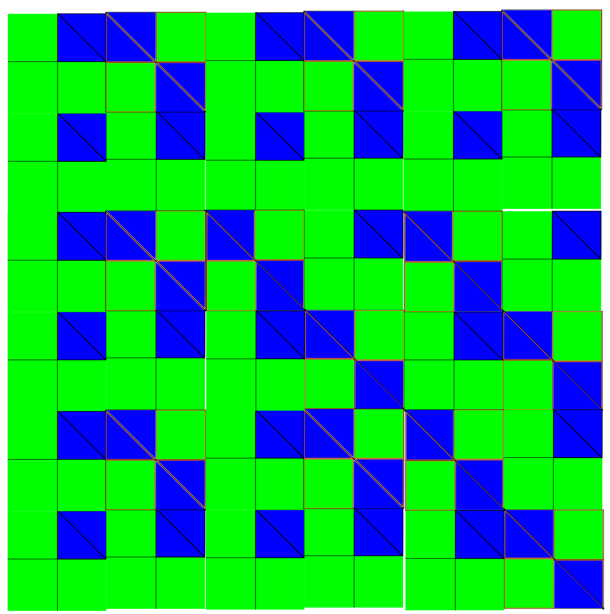


# of tiles in a patch corresponding to a vertex =
# of paths which reach that vertex





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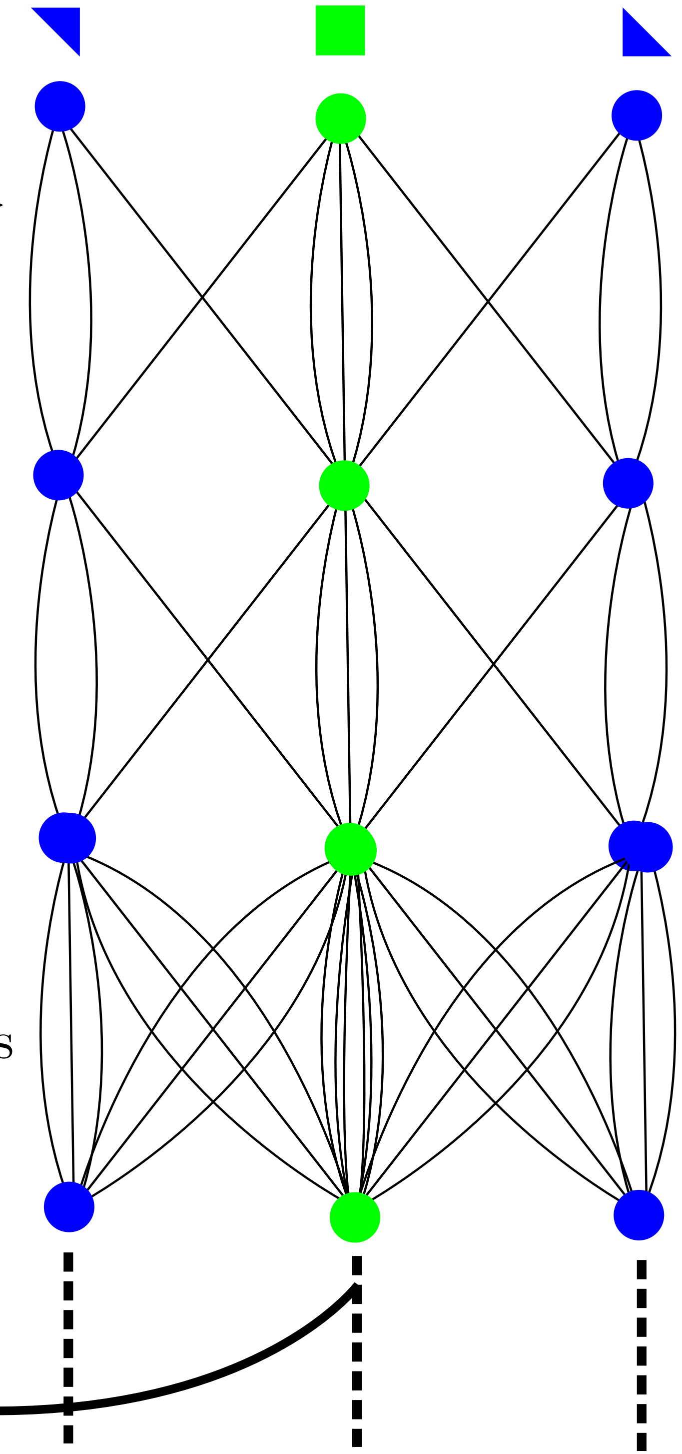

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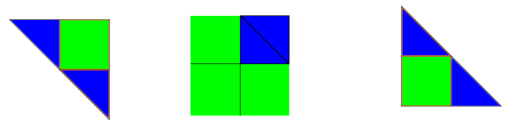

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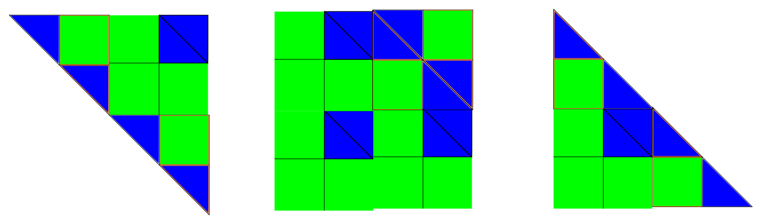
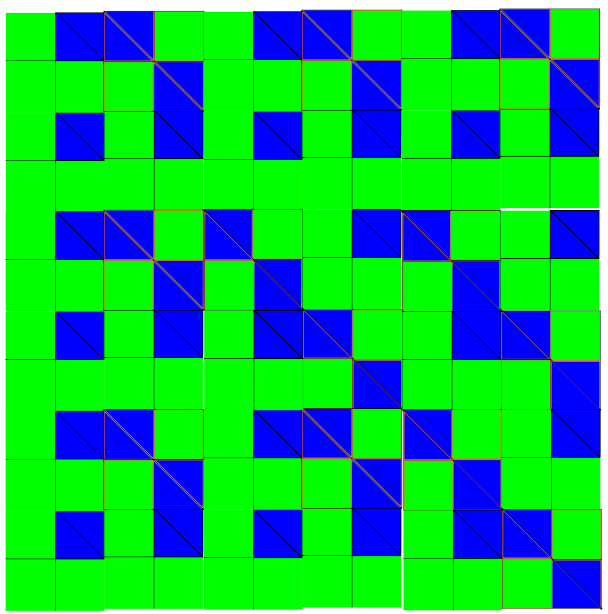
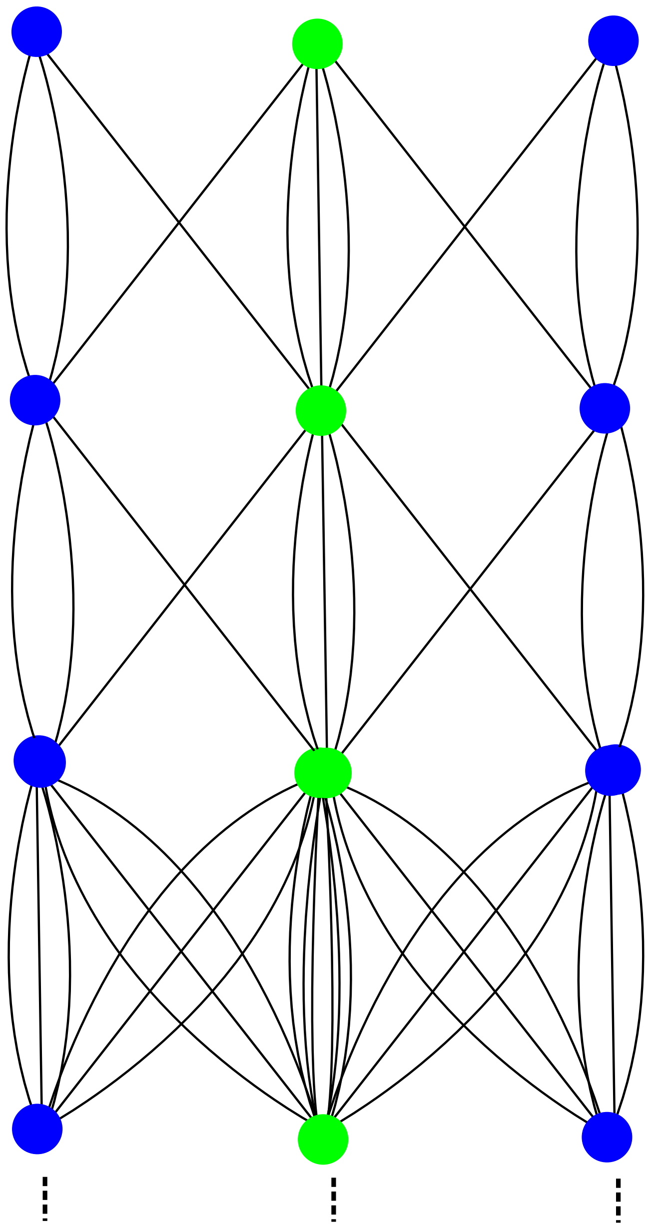


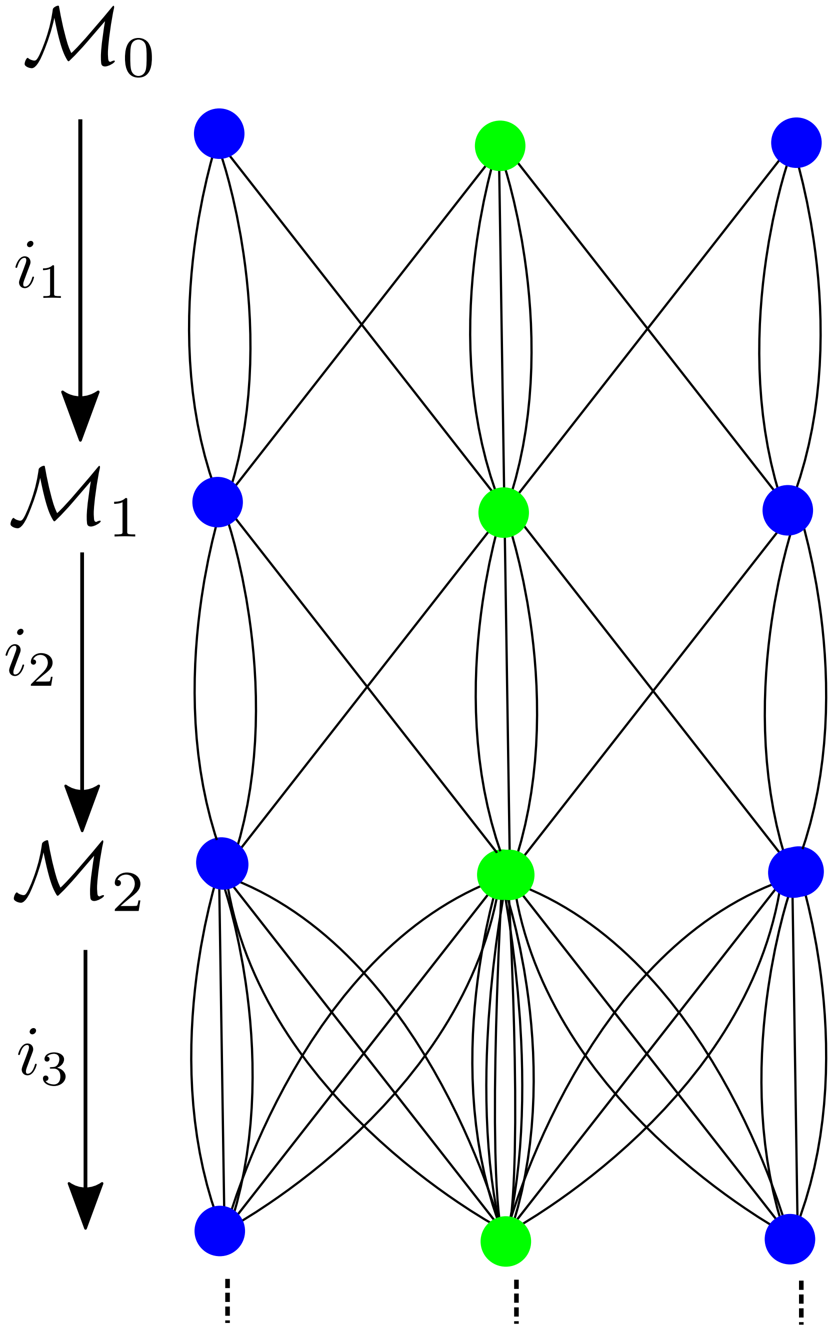


Tilings are obtained by taking limits of larger and larger patches.

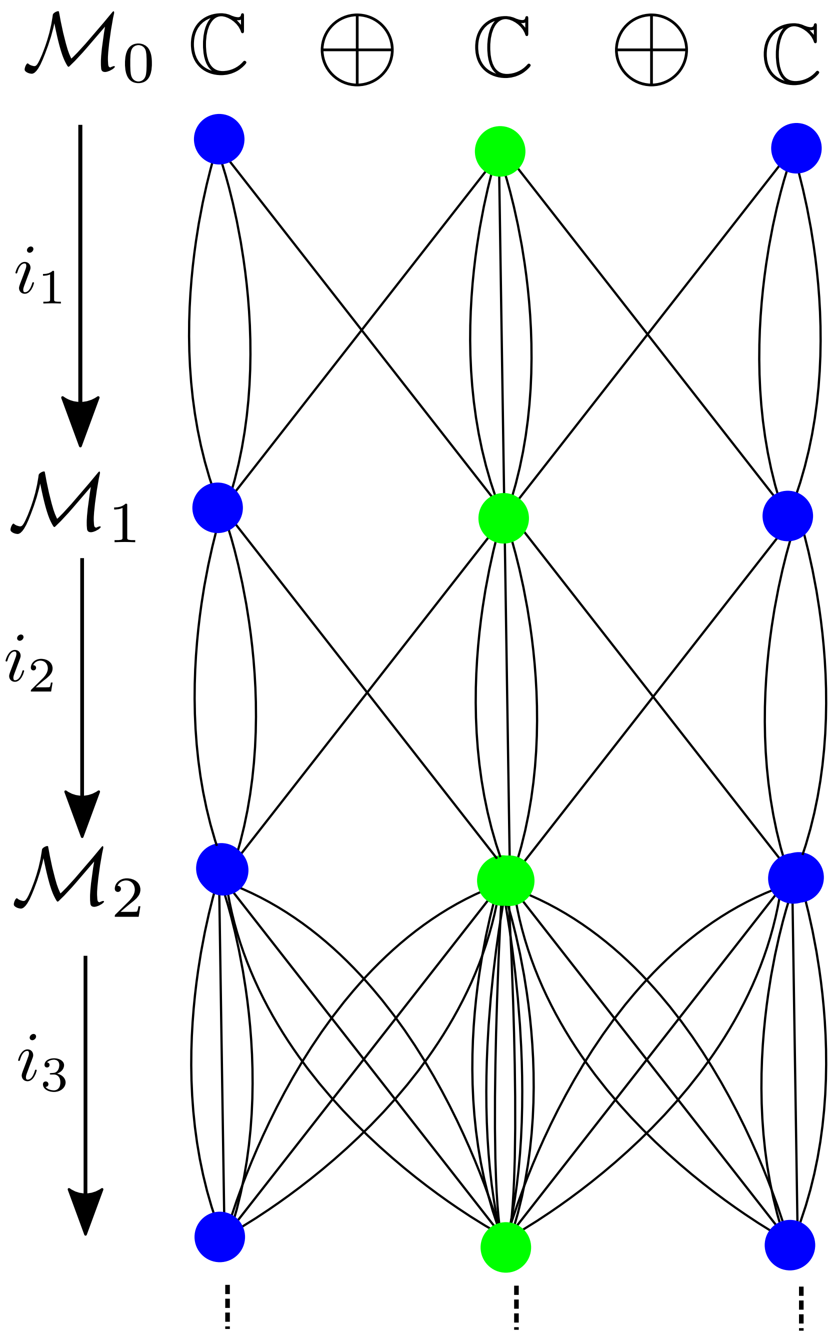
Bratteli diagram!



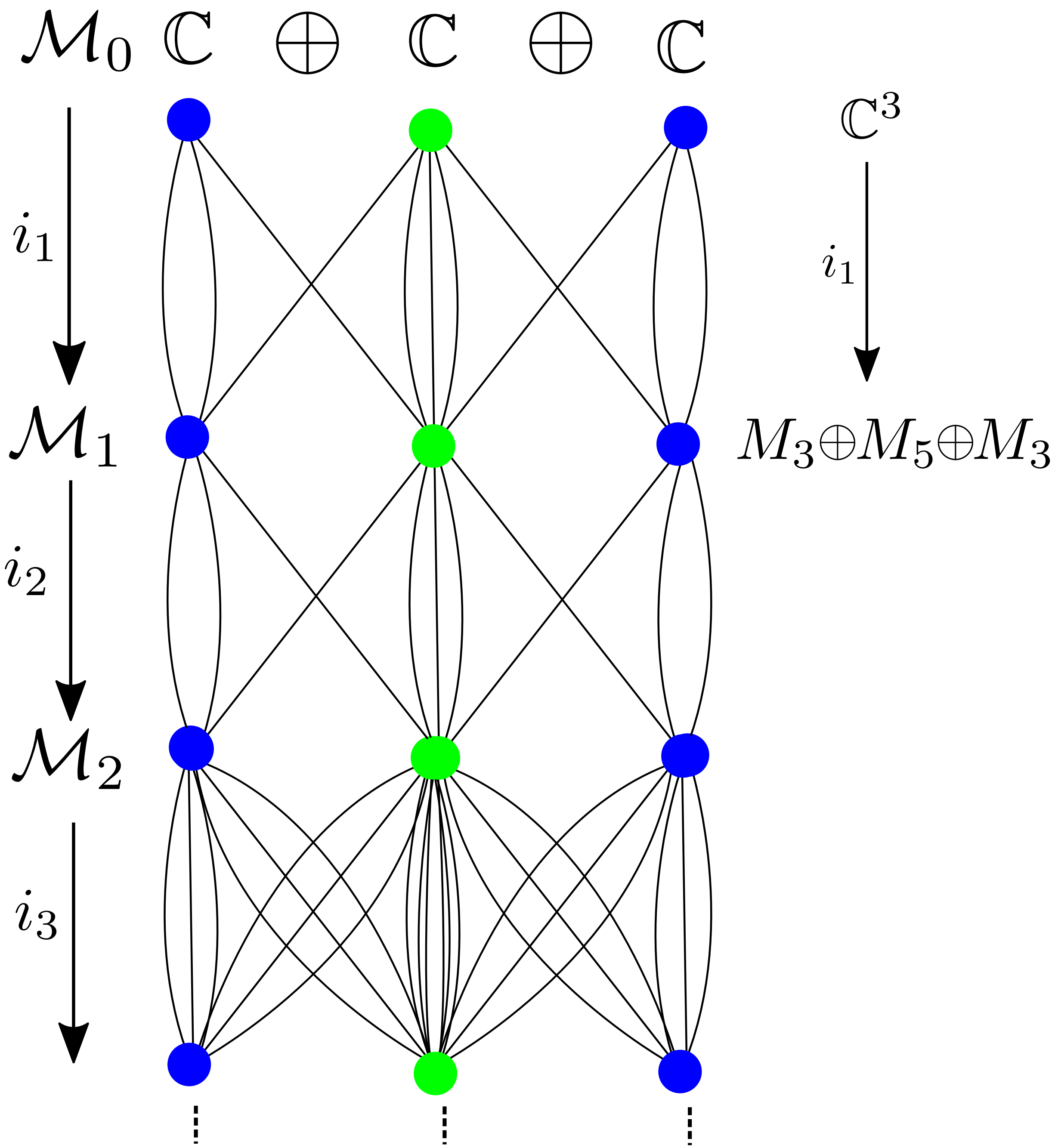
Bratteli diagram!



Bratteli diagram!

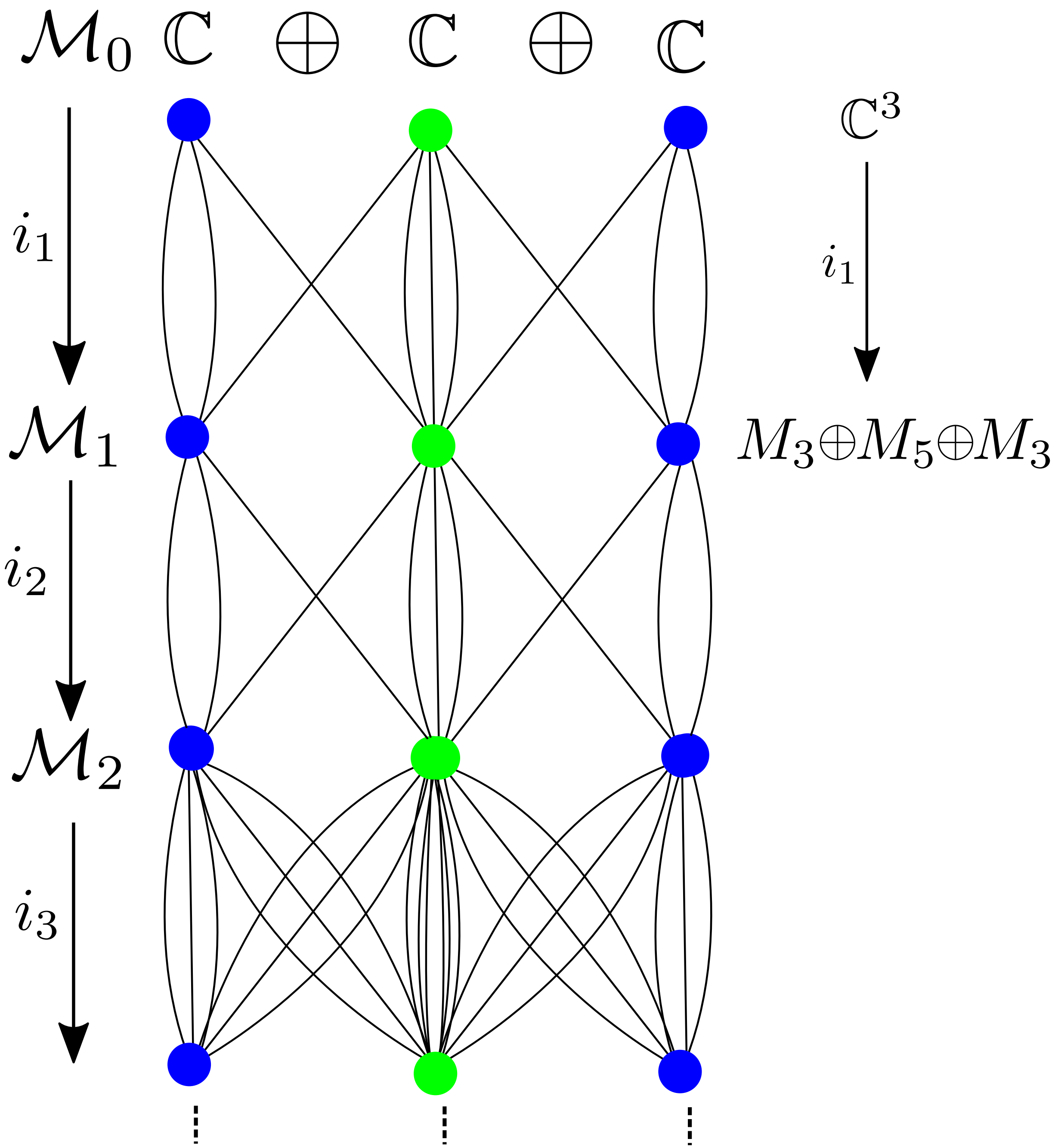


Bratteli diagram!

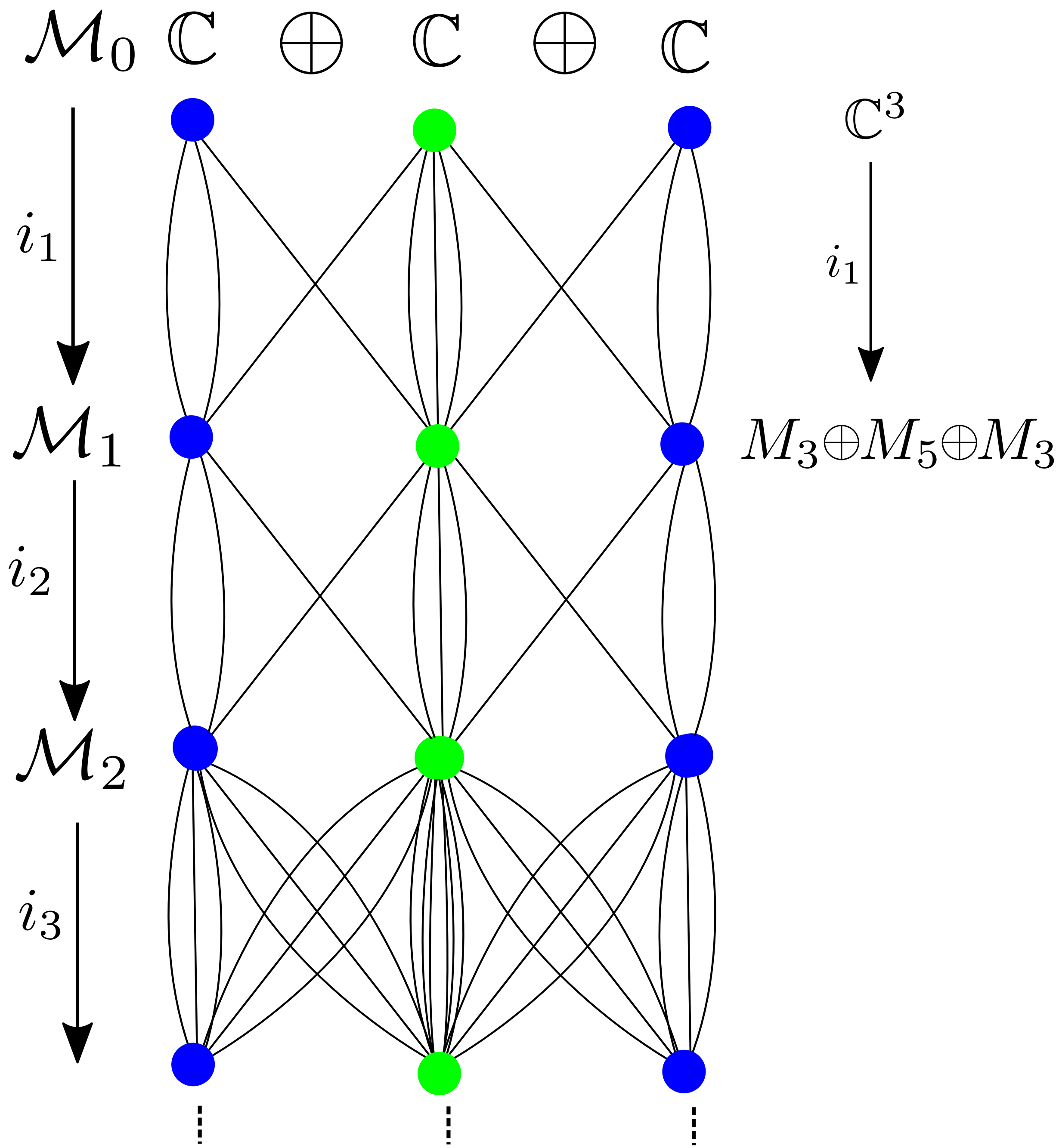


Bratteli diagram!

$(a, b, c) \in \mathbb{C}^3$



Bratteli diagram!

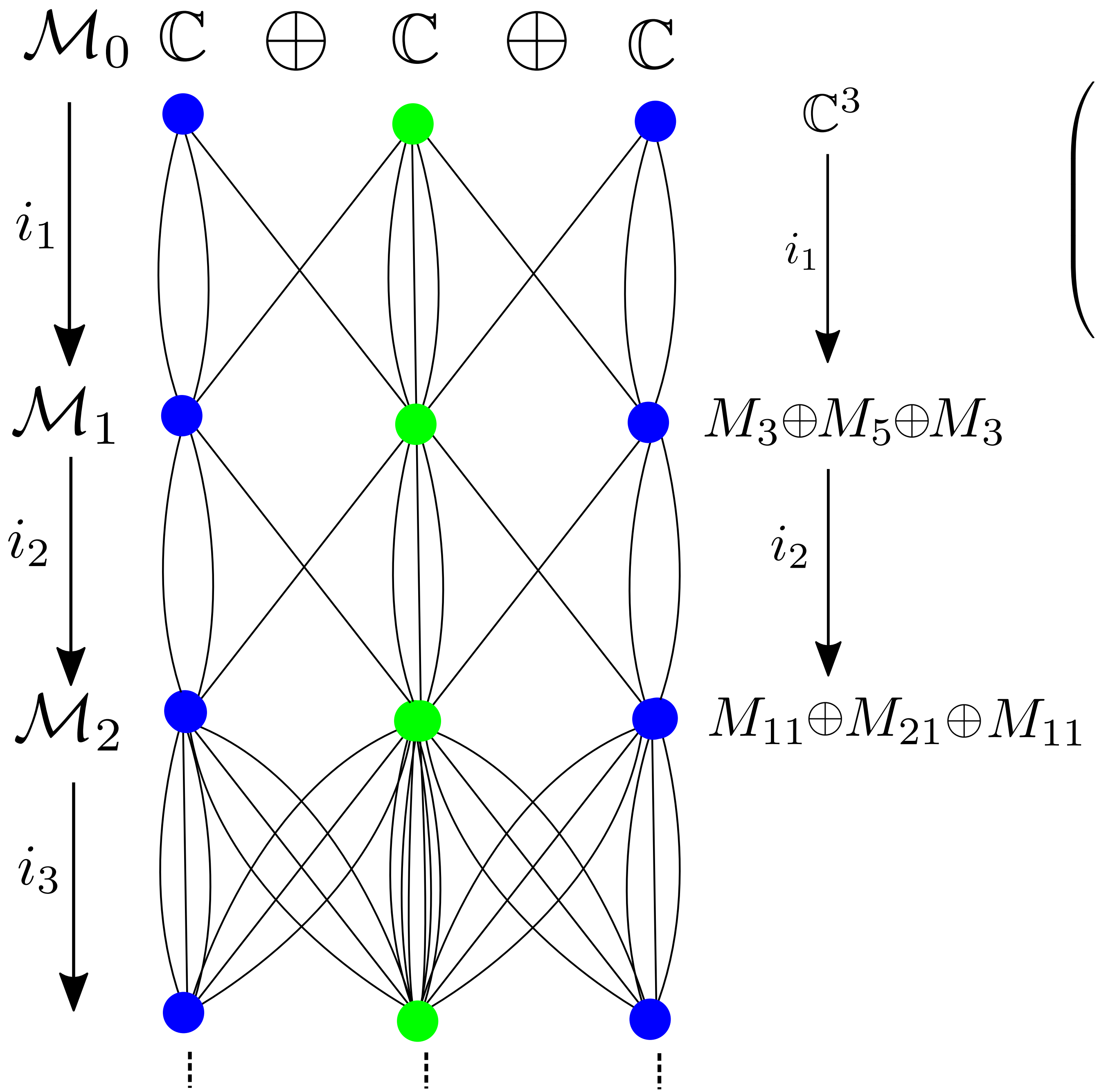


$$(a, b, c) \in \mathbb{C}^3$$

$$\Big\downarrow i_1$$

$$\left(\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & a \end{pmatrix}, \begin{pmatrix} c & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & b \end{pmatrix}, \begin{pmatrix} b & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{pmatrix} \right)$$

Bratteli diagram!



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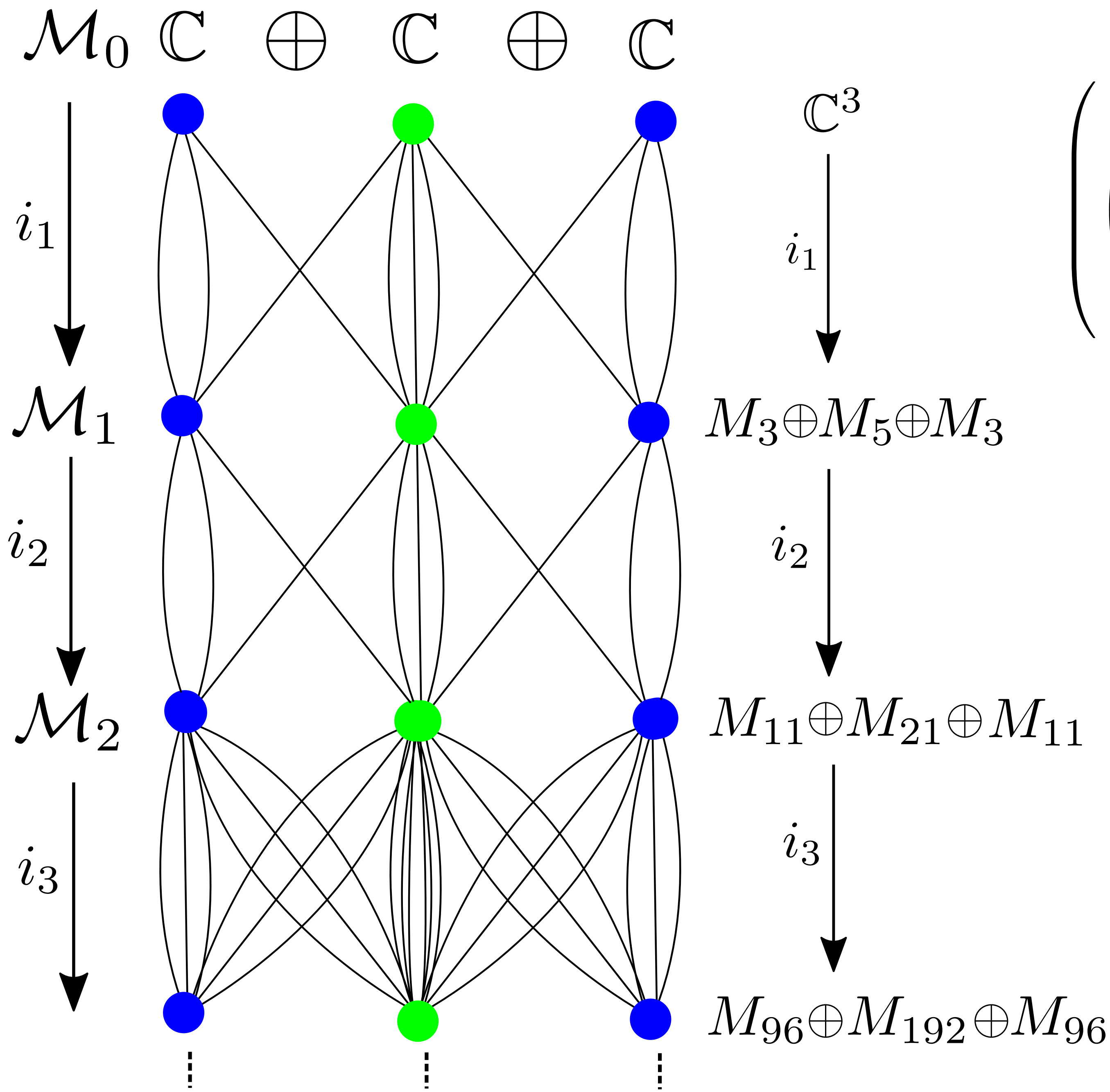

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$$\left(\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & a \end{pmatrix}, \begin{pmatrix} c & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & b \end{pmatrix}, \begin{pmatrix} b & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{pmatrix} \right)$$

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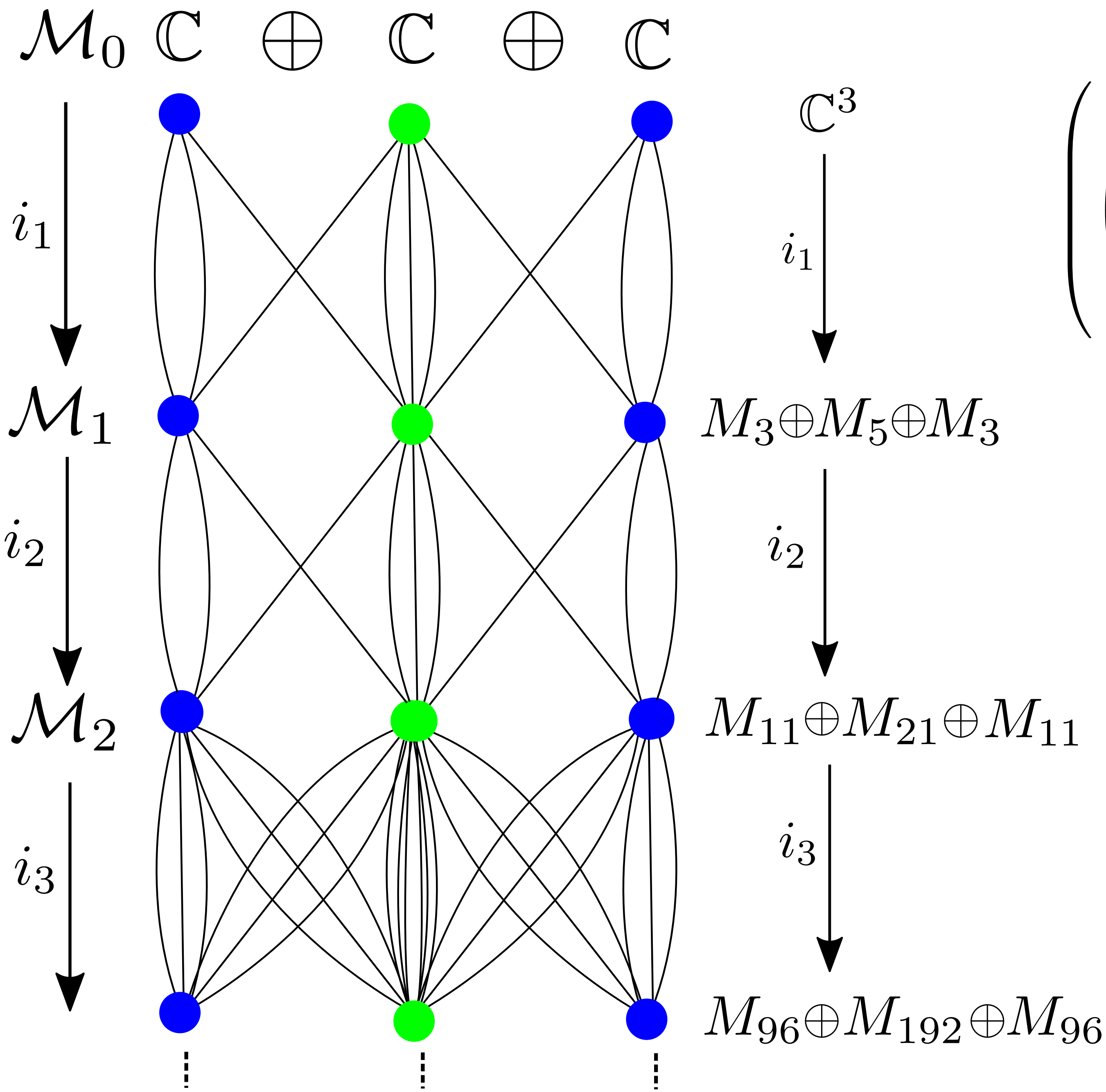

$$(a, b, c) \in \mathbb{C}^3$$

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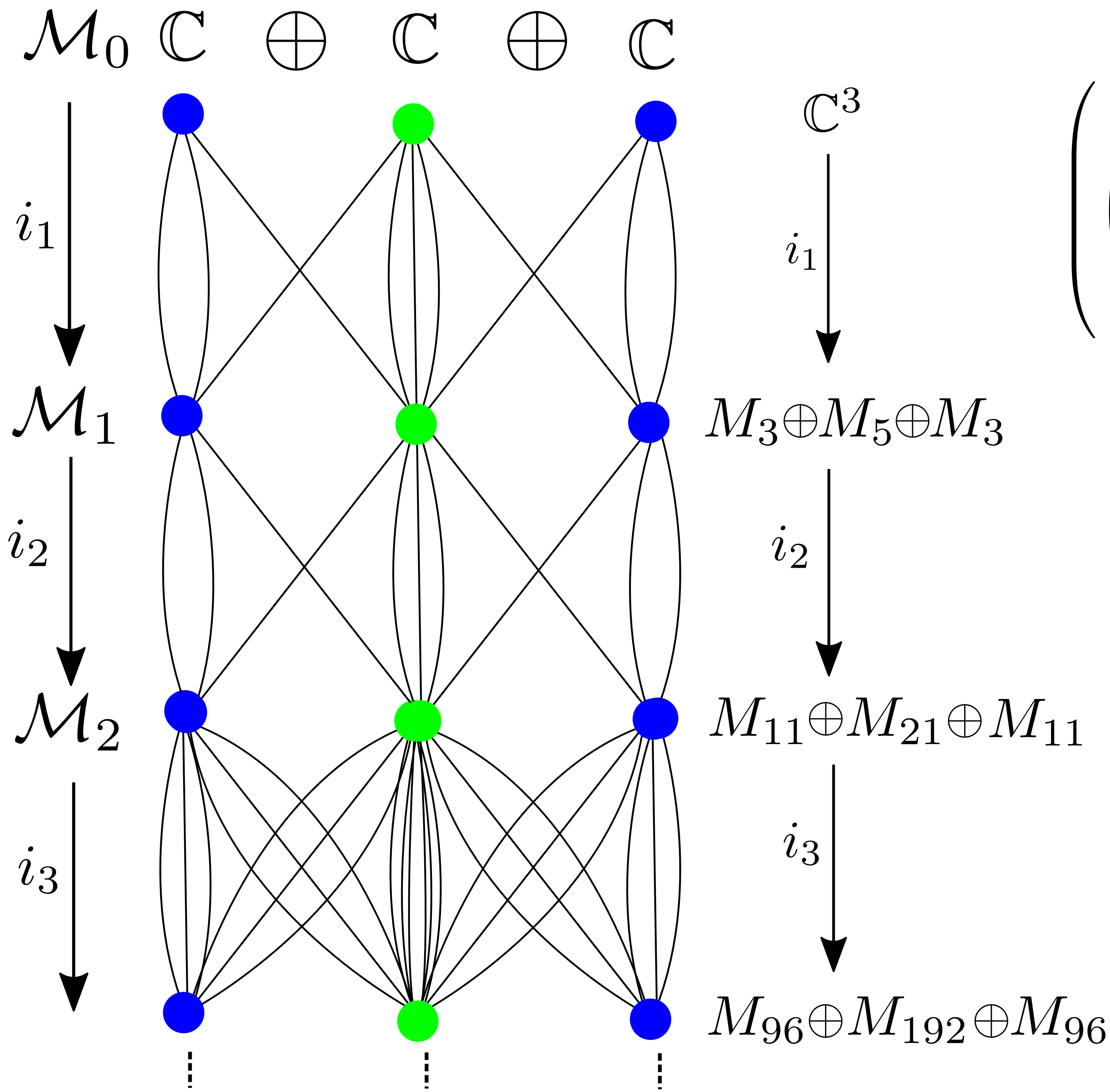
$$\left(\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & a \end{pmatrix}, \begin{pmatrix} c & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & b \end{pmatrix}, \begin{pmatrix} b & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{pmatrix} \right)$$

$$LF(B) = \varinjlim (\mathcal{M}_k, i_k)$$

Bratteli diagram!



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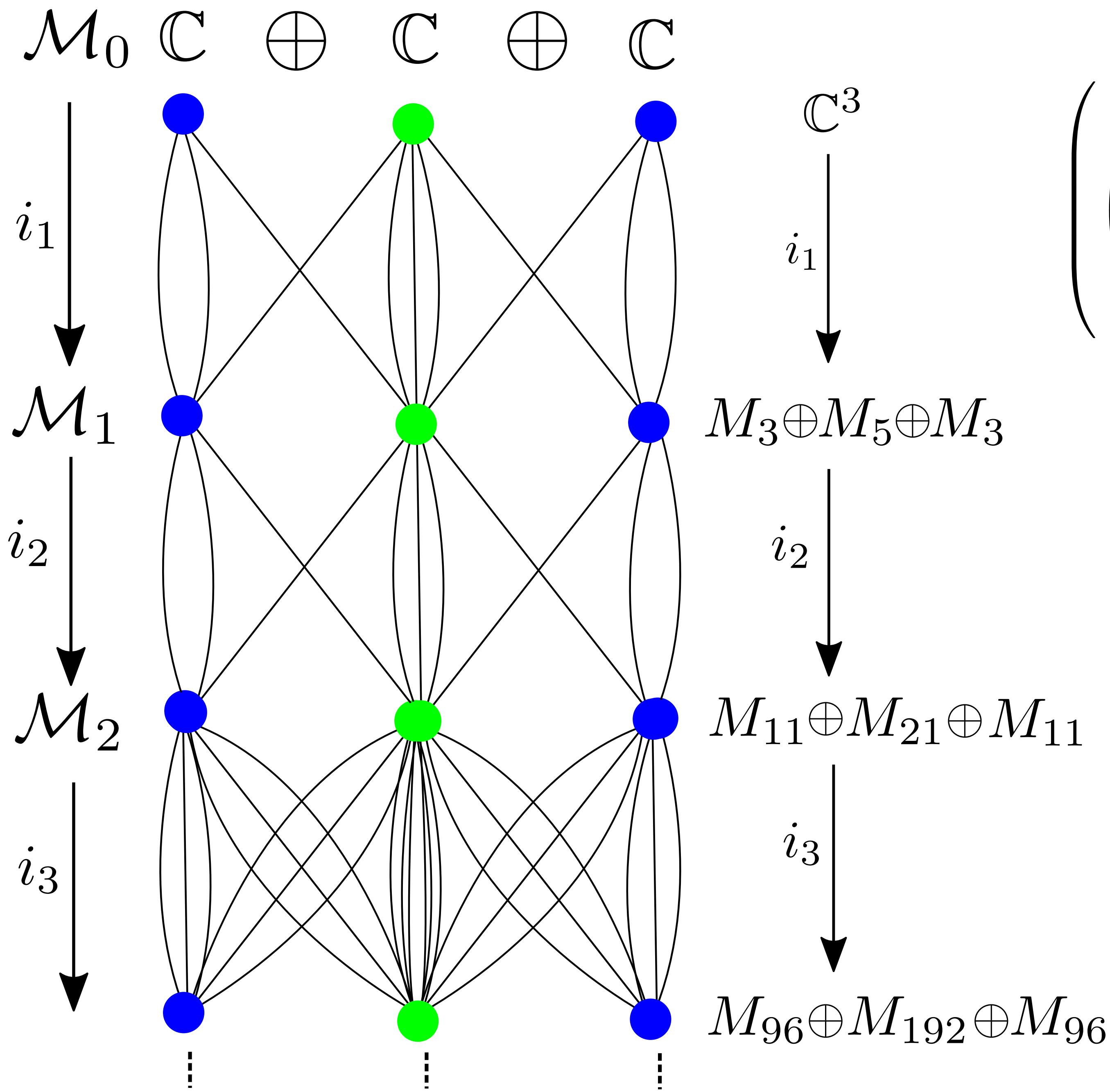
Locally finite algebra

Bratteli diagram!



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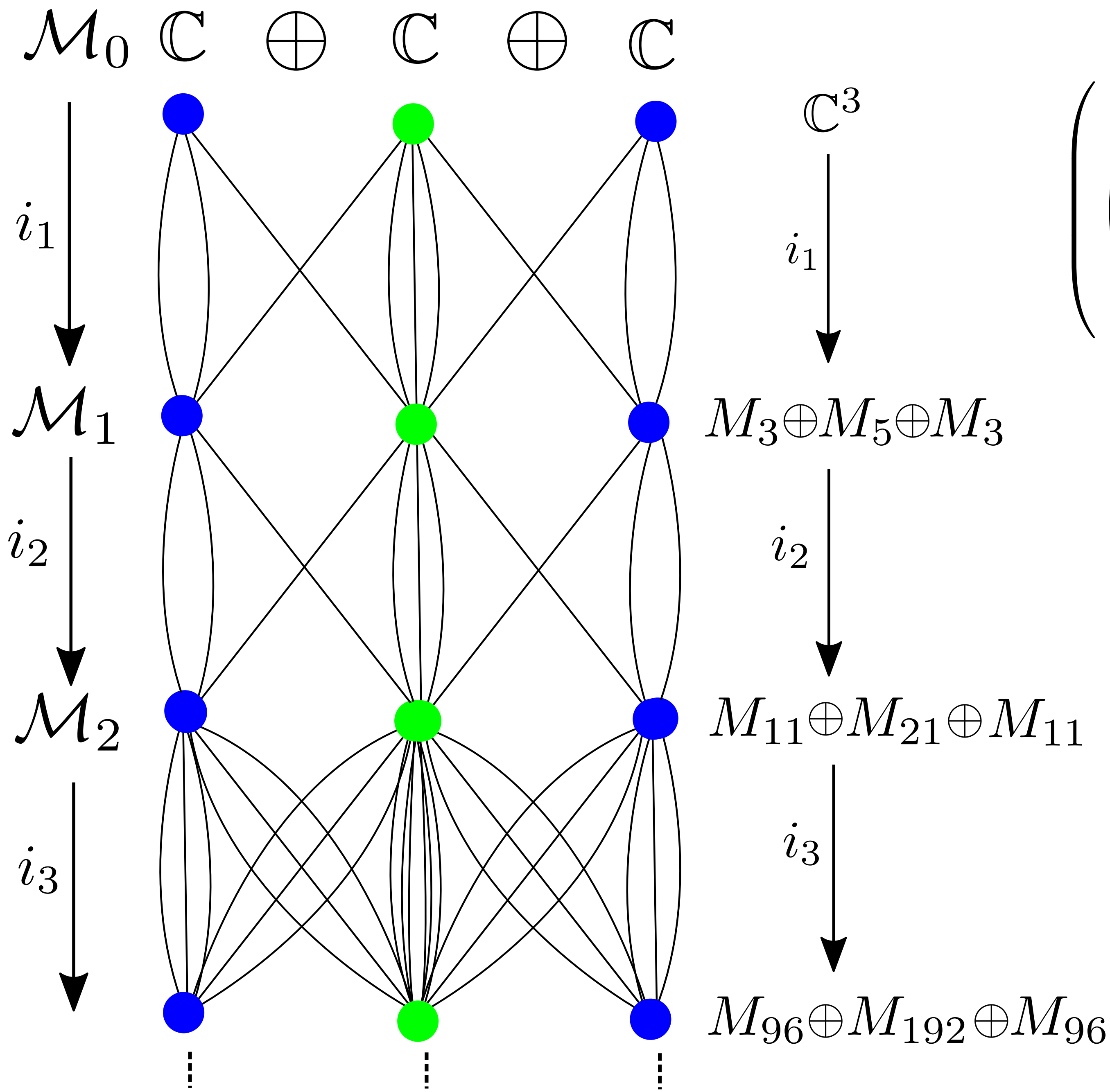
$$LF(B) = \varinjlim (\mathcal{M}_k, i_k)$$

Locally finite algebra

If $\mathcal{M}_k = \bigoplus_{i=1}^{\ell} M_{n_i}$ then $\mathcal{M}_k^* = \mathrm{Tr}(\mathcal{M}_k) \cong \mathbb{C}^\ell$

Bratteli diagram!



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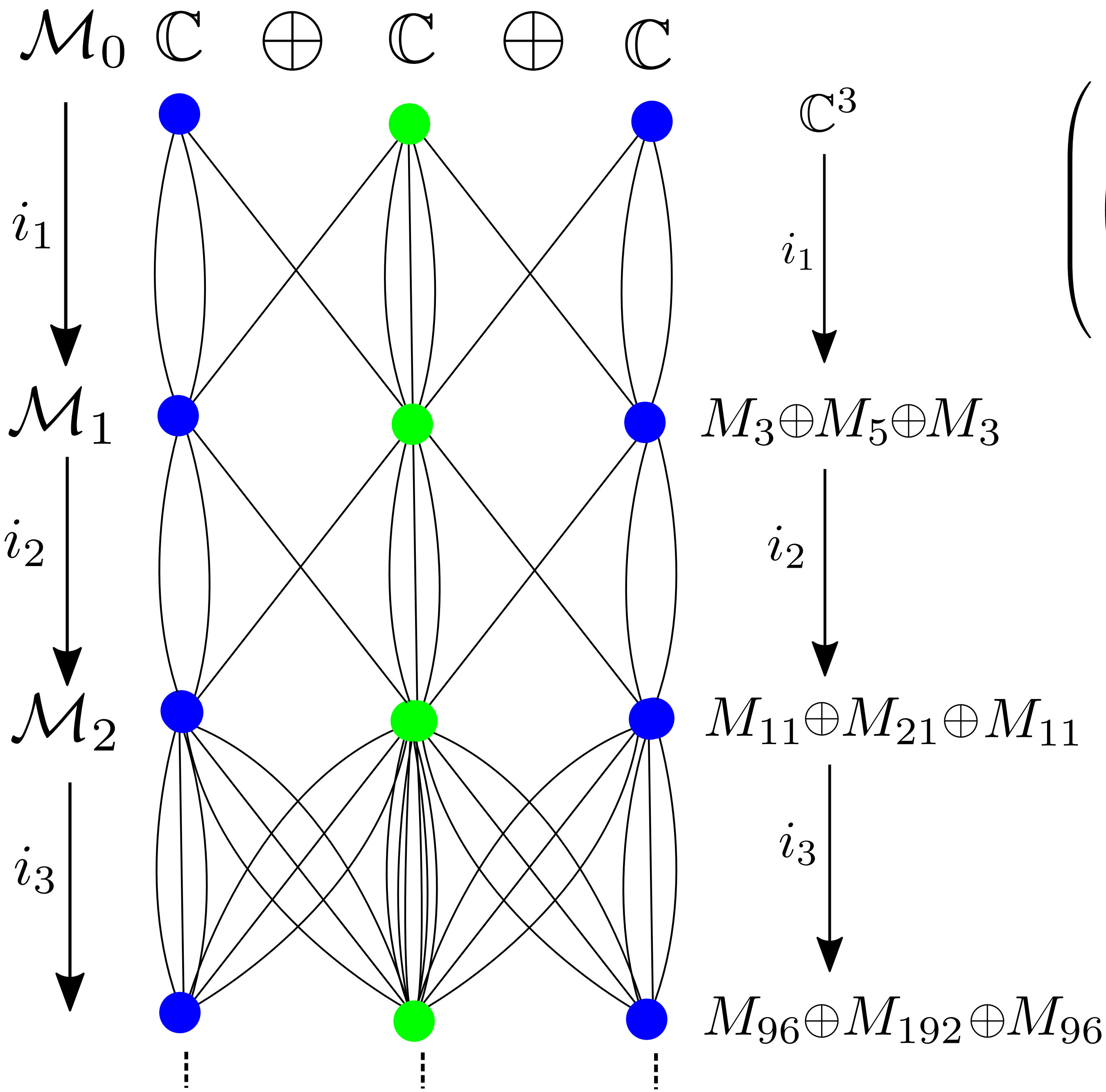
Locally finite algebra

If $\mathcal{M}_k = \bigoplus_{i=1}^{\ell} M_{n_i}$ then $\mathcal{M}_k^* = \mathrm{Tr}(\mathcal{M}_k) \cong \mathbb{C}^\ell$

Trace space

Bratteli diagram!



$$(a, b, c) \in \mathbb{C}^3$$

$$\downarrow i_1$$

$$\left(\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & a \end{pmatrix}, \begin{pmatrix} c & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & b \end{pmatrix}, \begin{pmatrix} b & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{pmatrix} \right)$$

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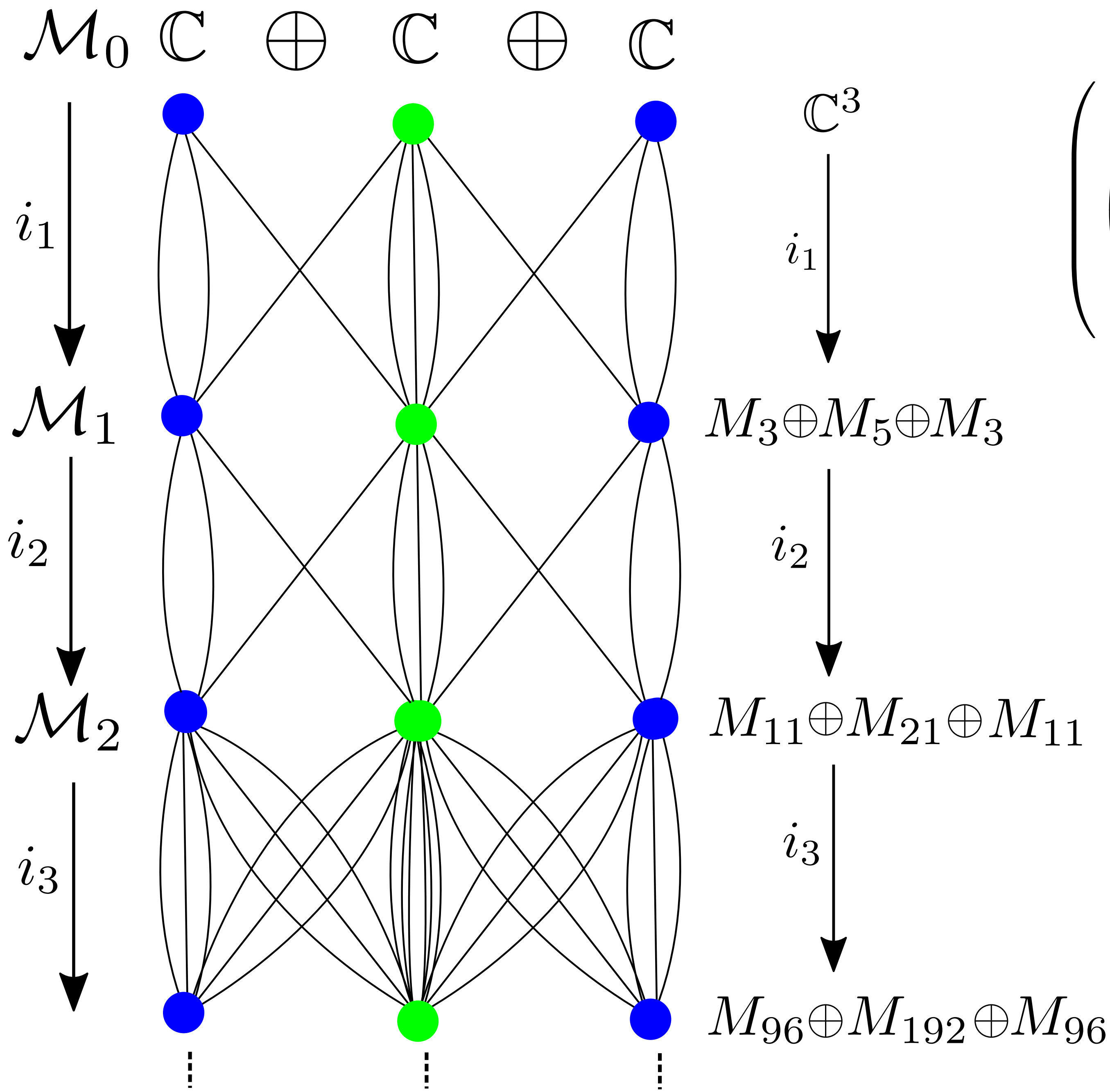
Locally finite algebra

If $\mathcal{M}_k = \bigoplus_{i=1}^{\ell} M_{n_i}$ then $\mathcal{M}_k^* = \mathrm{Tr}(\mathcal{M}_k) \cong \mathbb{C}^\ell$

Trace space

$$LF(B)^* = \mathrm{Tr}(B) = \varprojlim (\mathrm{Tr}(\mathcal{M}_k), i_k^*)$$

Bratteli diagram!



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$$LF(B) = \varinjlim (\mathcal{M}_k, i_k)$$

Locally finite algebra

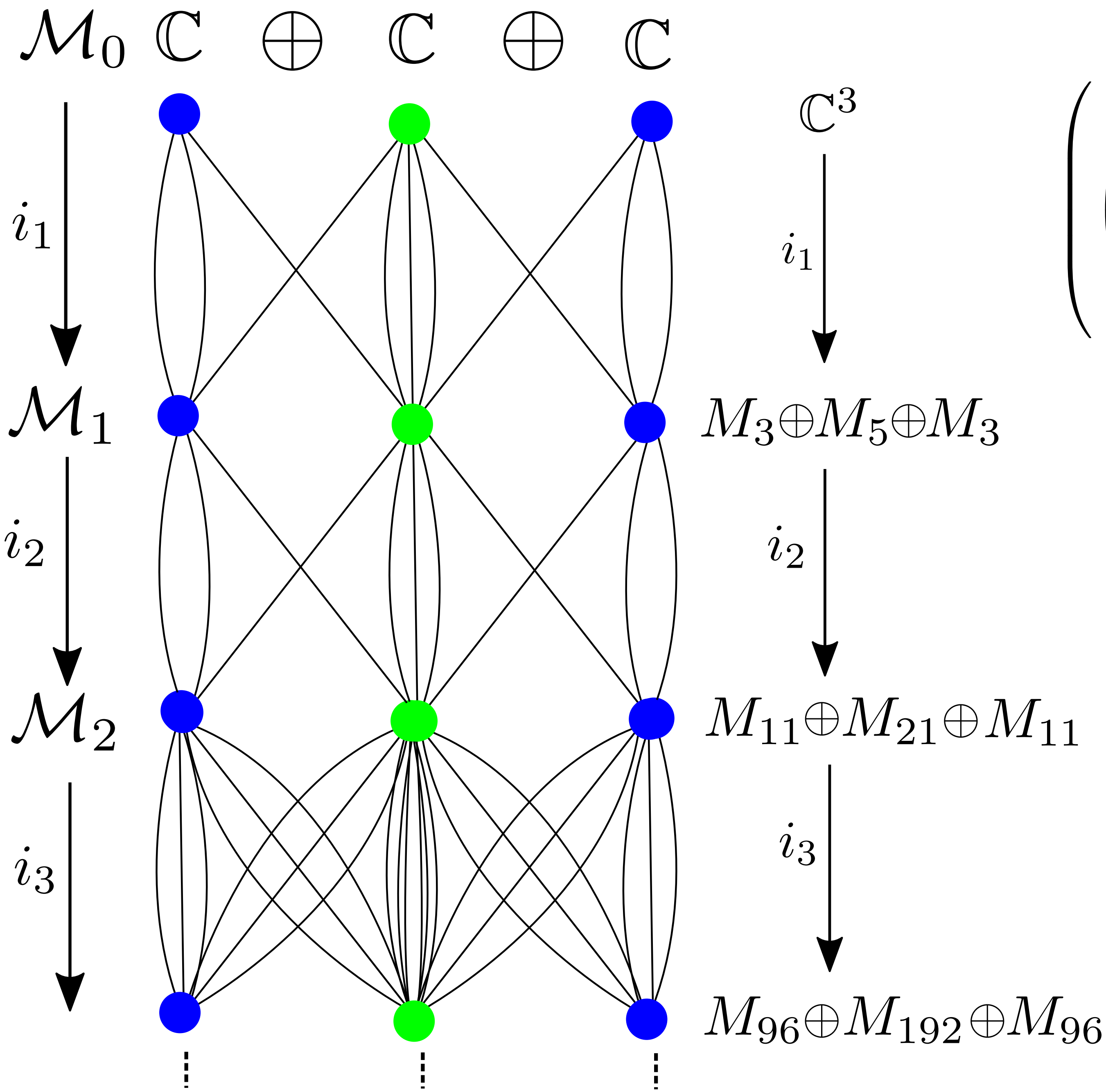
If $\mathcal{M}_k = \bigoplus_{i=1}^{\ell} M_{n_i}$ then $\mathcal{M}_k^* = \mathrm{Tr}(\mathcal{M}_k) \cong \mathbb{C}^\ell$

Trace space

$$LF(B)^* = \mathrm{Tr}(B) = \varprojlim (\mathrm{Tr}(\mathcal{M}_k), i_k^*)$$

$$(= K_0(AF(B))^*)$$

Bratteli diagram!



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$$LF(B) = \varinjlim (\mathcal{M}_k, i_k)$$

Locally finite algebra

If $\mathcal{M}_k = \bigoplus_{i=1}^{\ell} M_{n_i}$ then $\mathcal{M}_k^* = \mathrm{Tr}(\mathcal{M}_k) \cong \mathbb{C}^\ell$

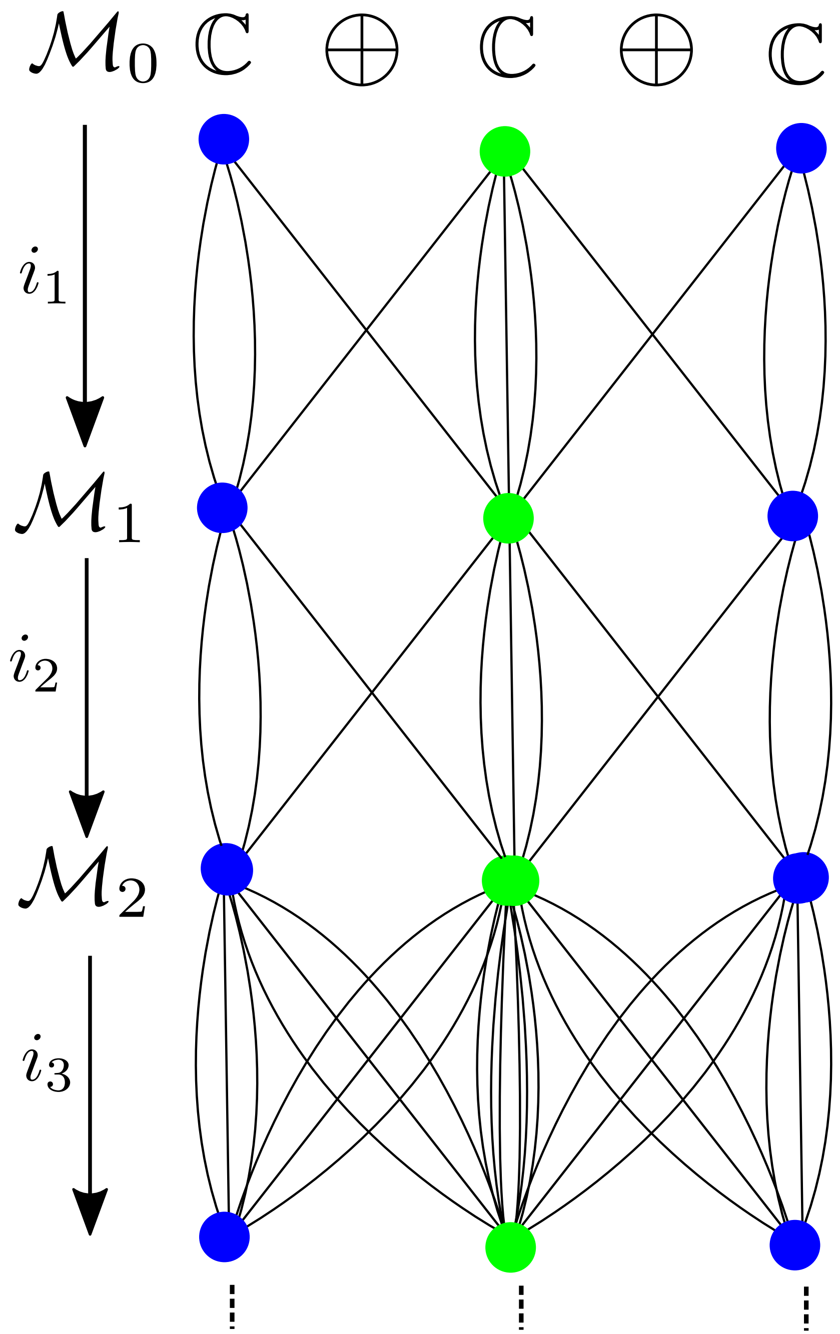
Trace space

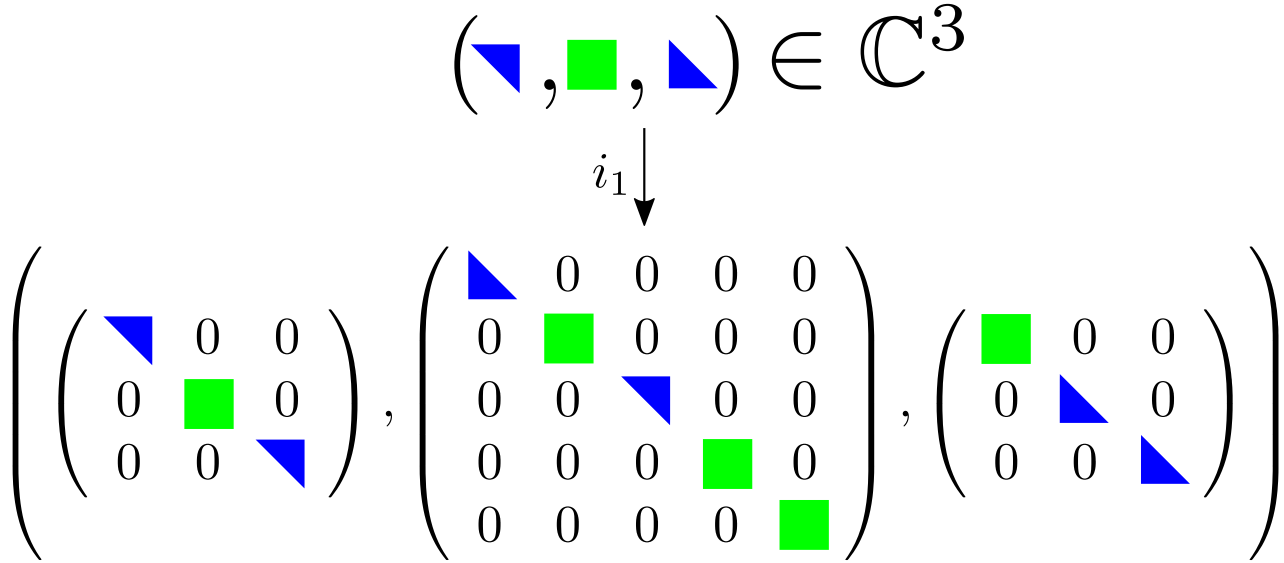
$$LF(B)^* = \mathrm{Tr}(B) = \varprojlim (\mathrm{Tr}(\mathcal{M}_k), i_k^*)$$

$$(= K_0(AF(B))^*)$$

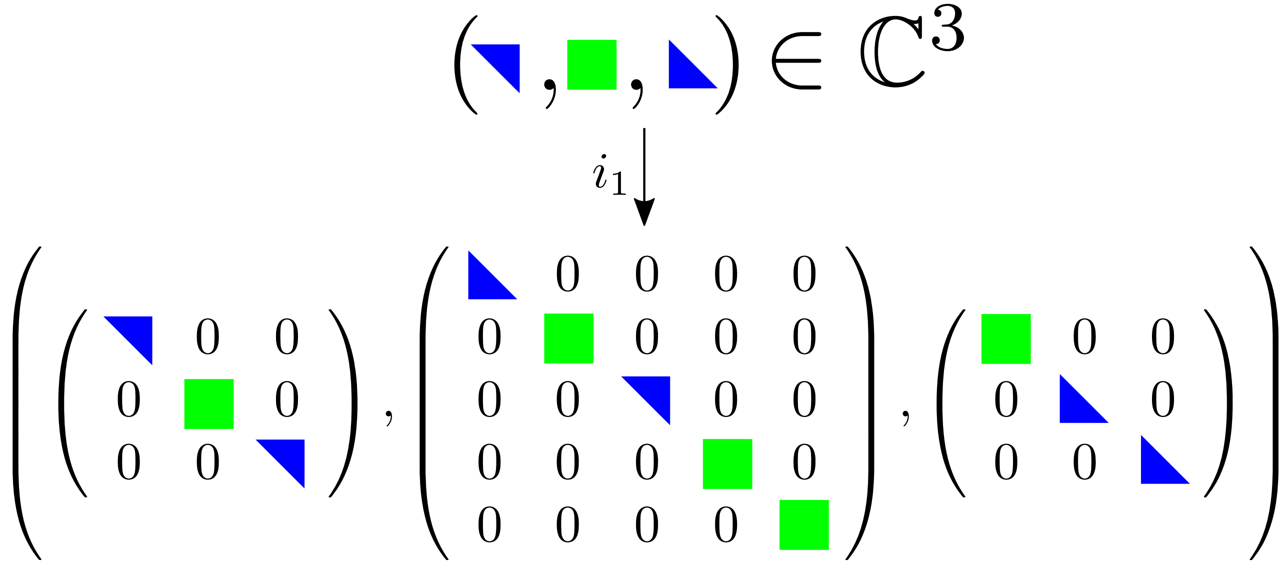
The maps i_k^* are given by the adjacency (i.e. substitution) matrix

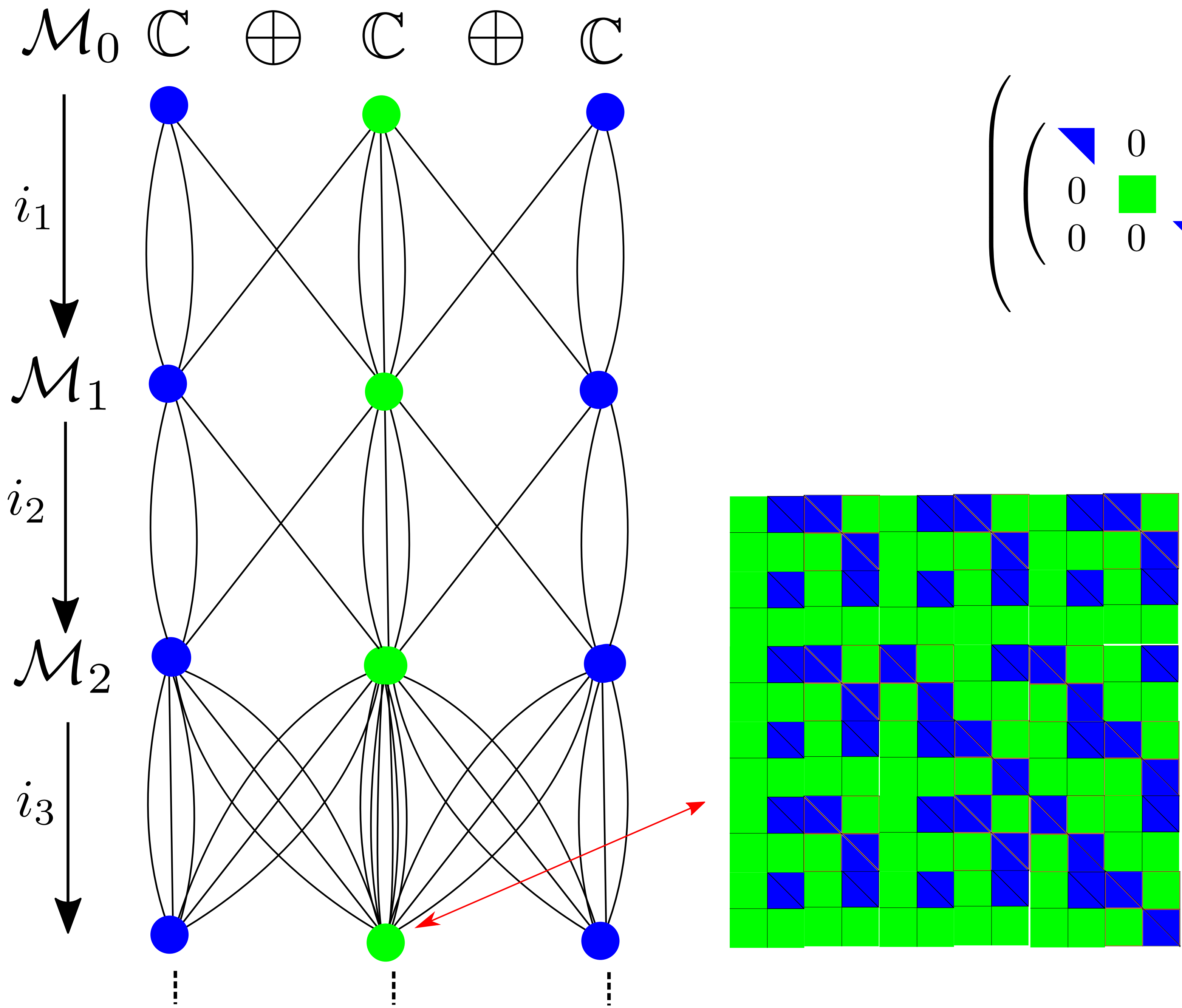
Bratteli diagram!



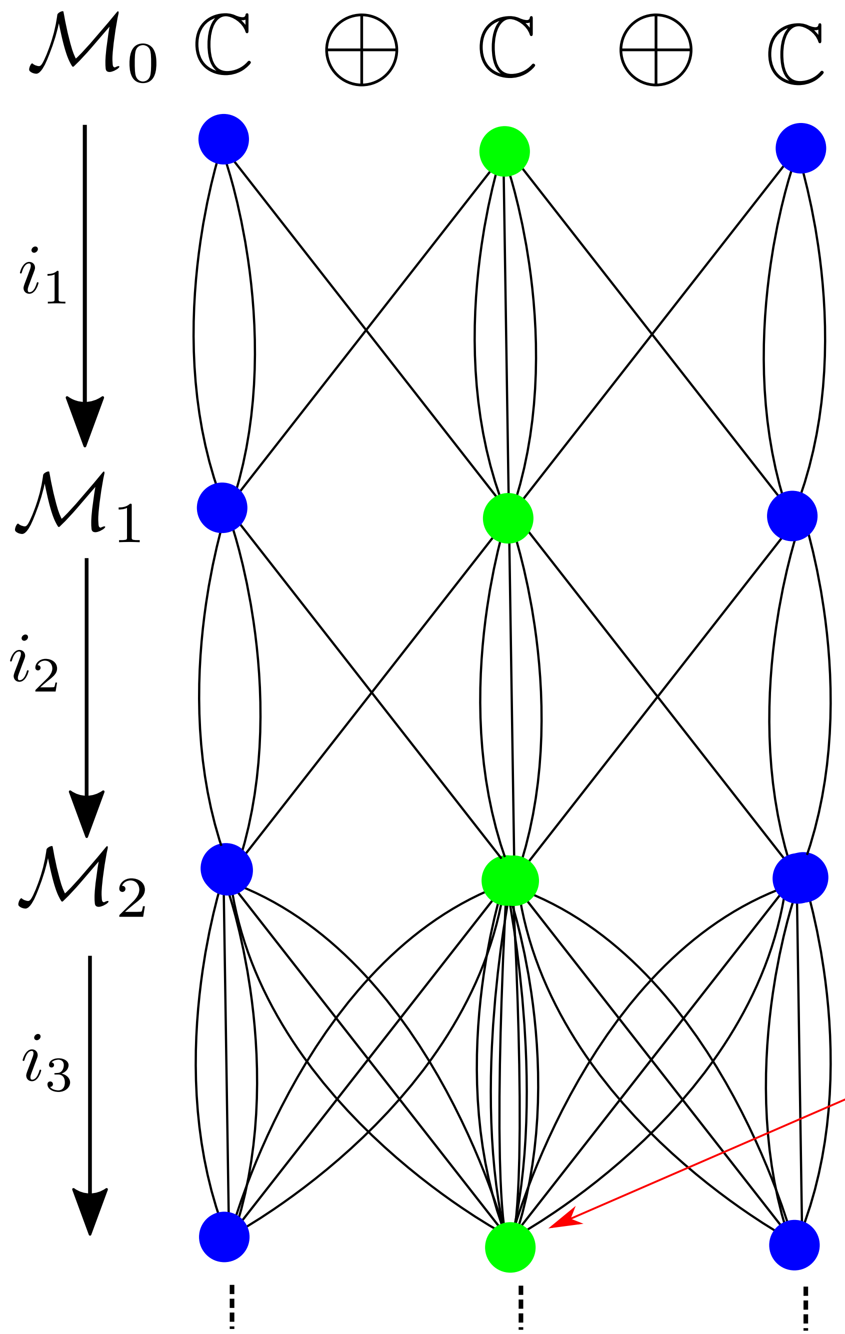


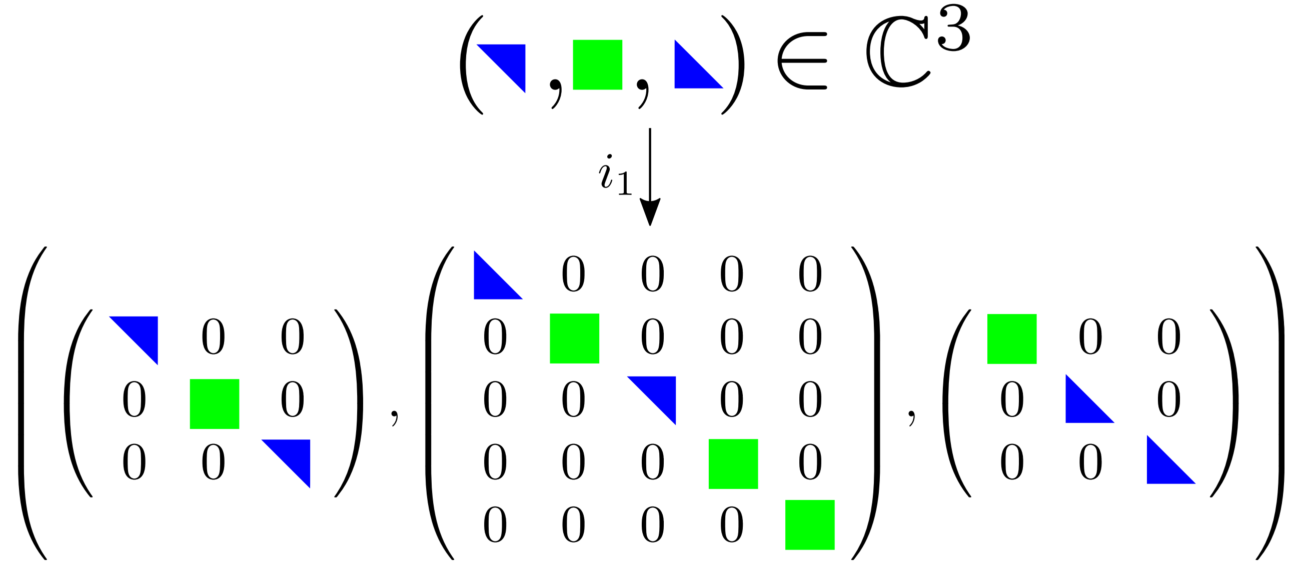
Bratteli diagram!



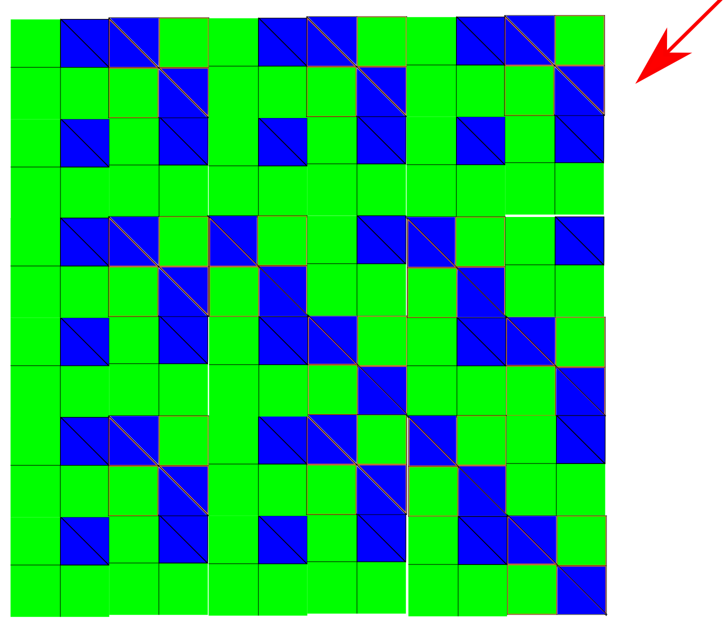


Bratteli diagram!

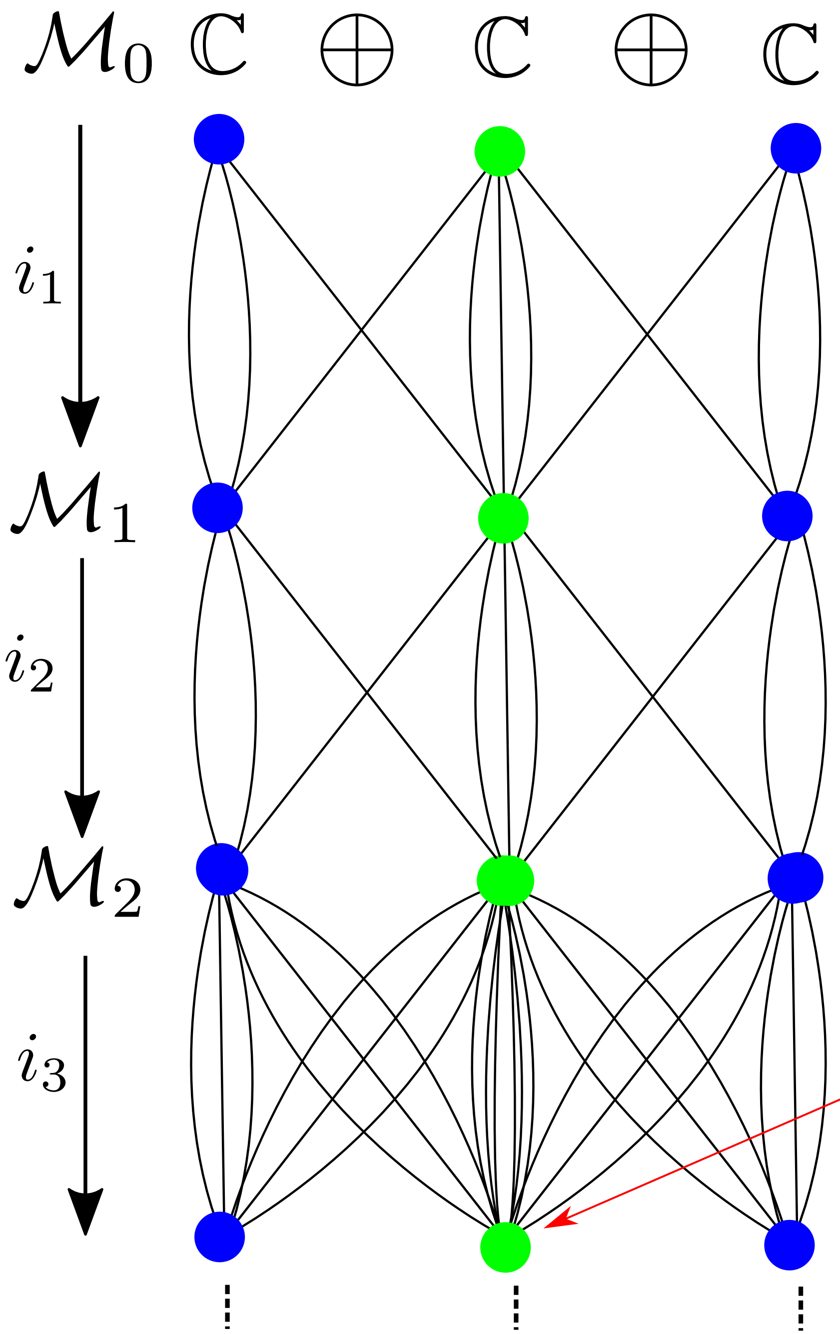


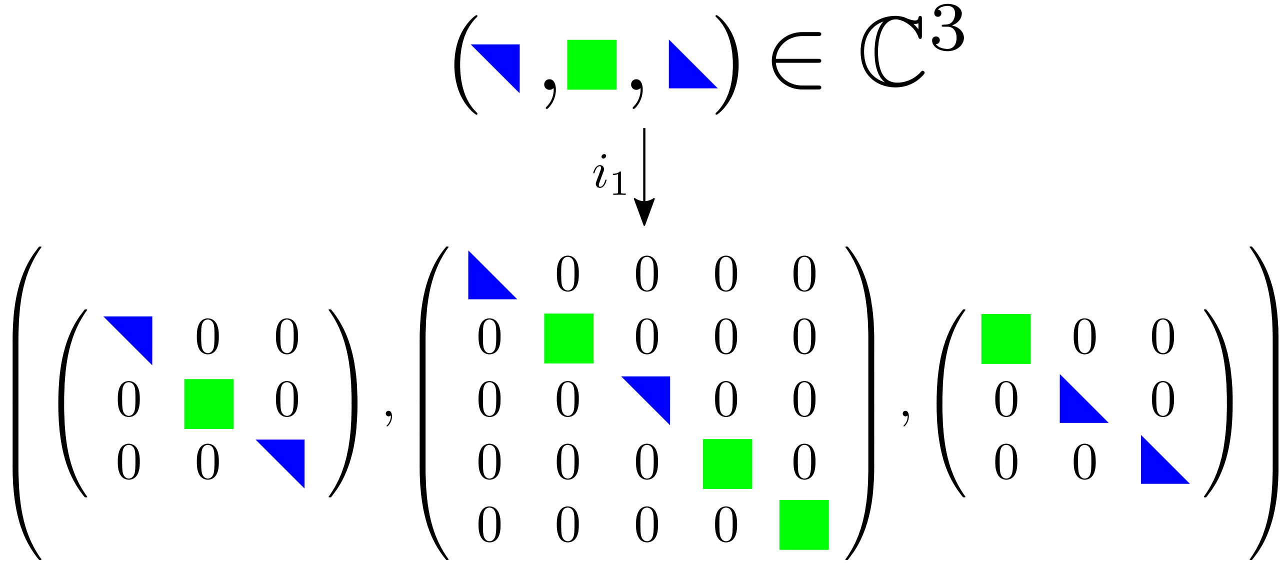


How many ■ do I see in ?



Bratteli diagram!





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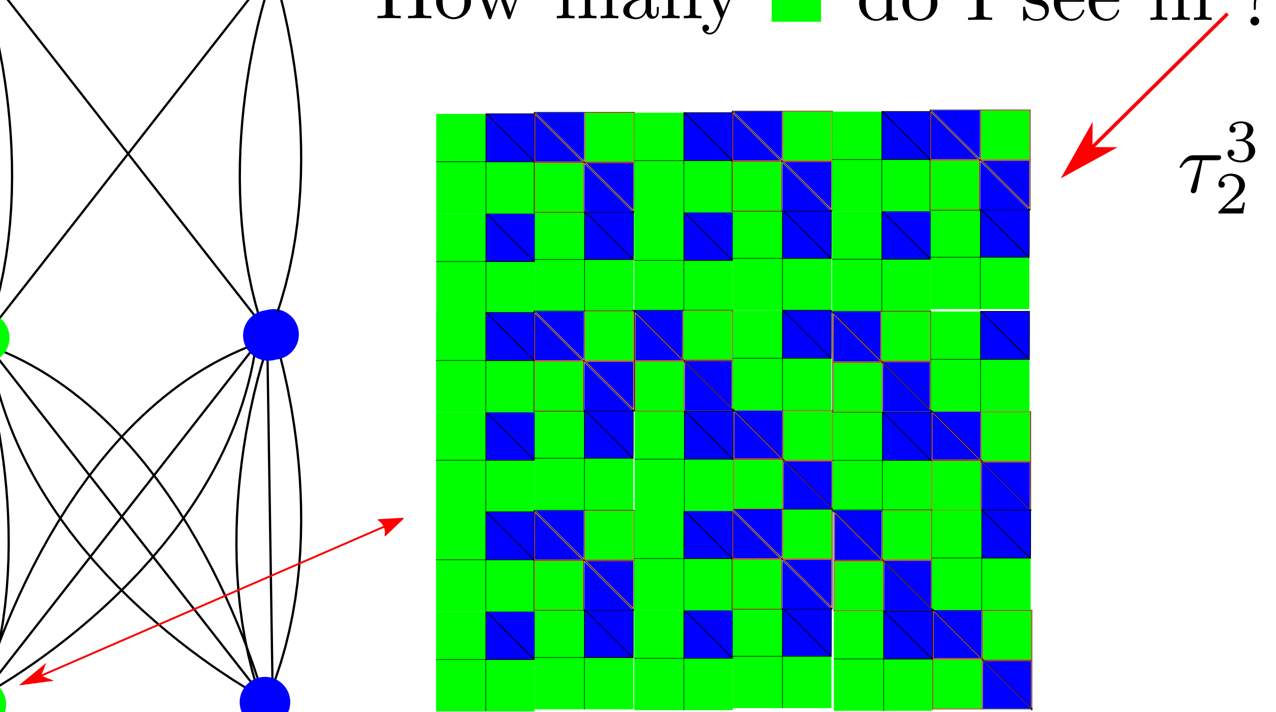
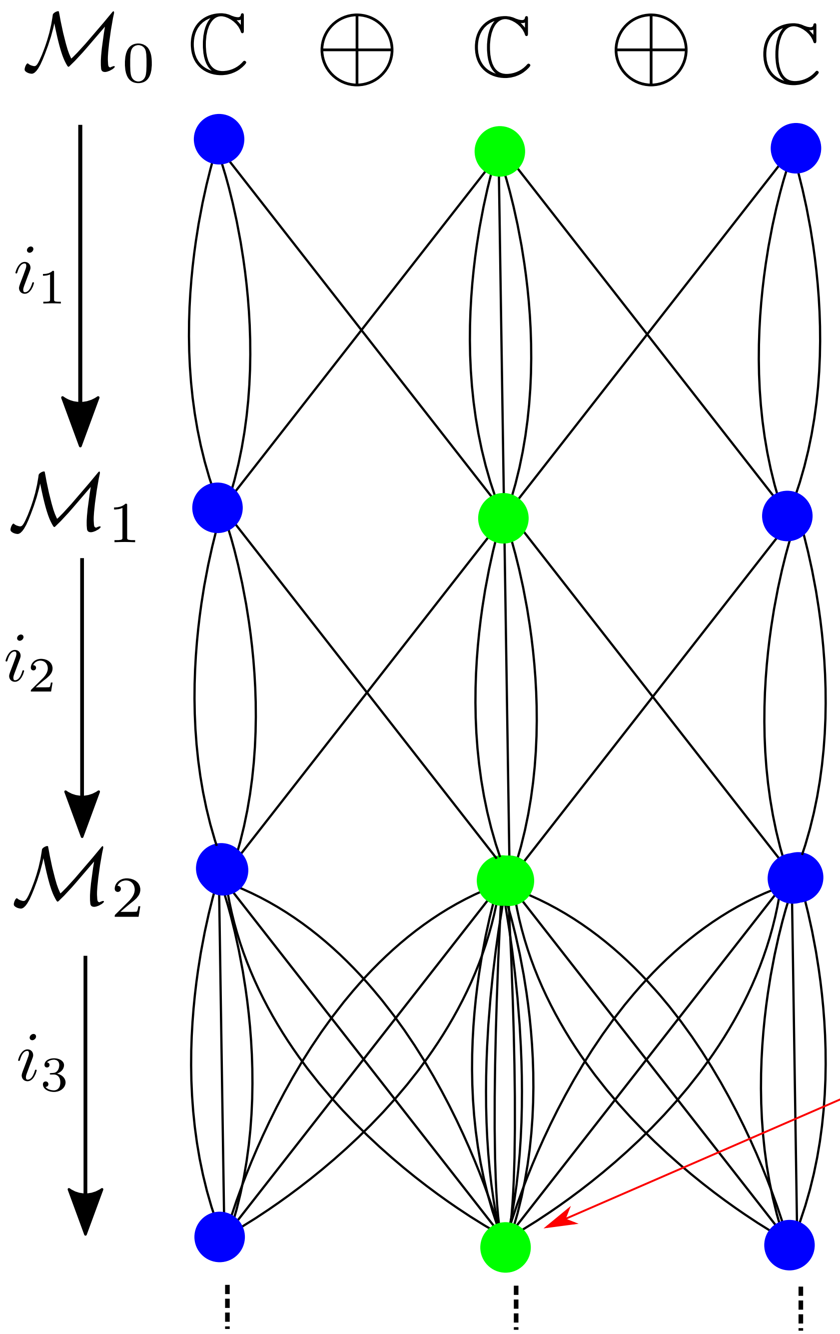
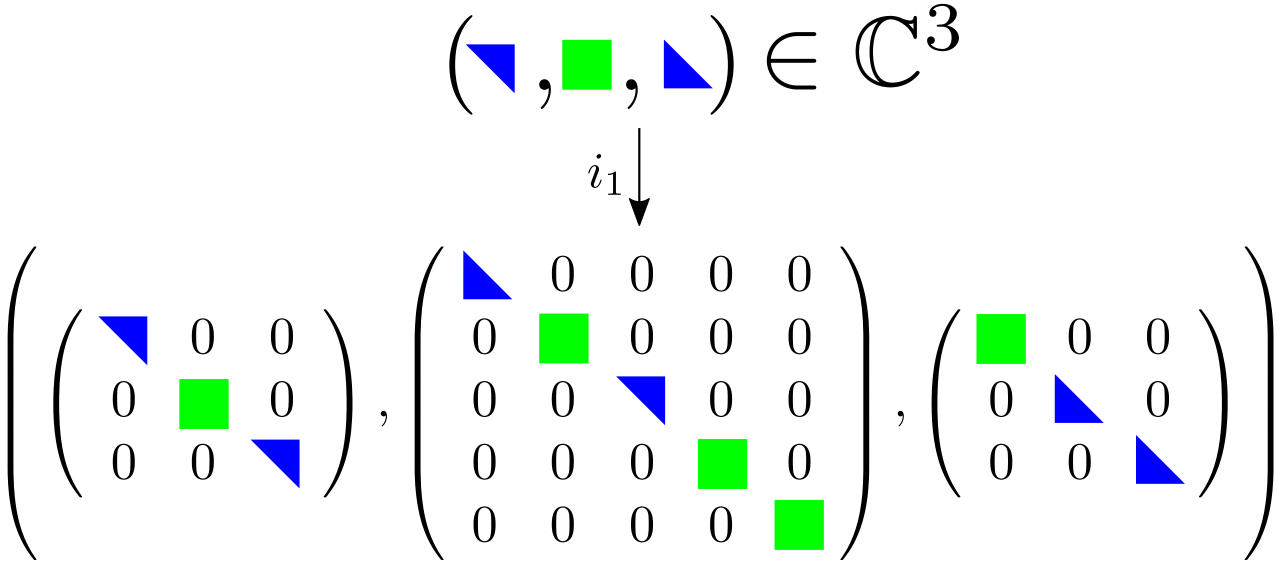


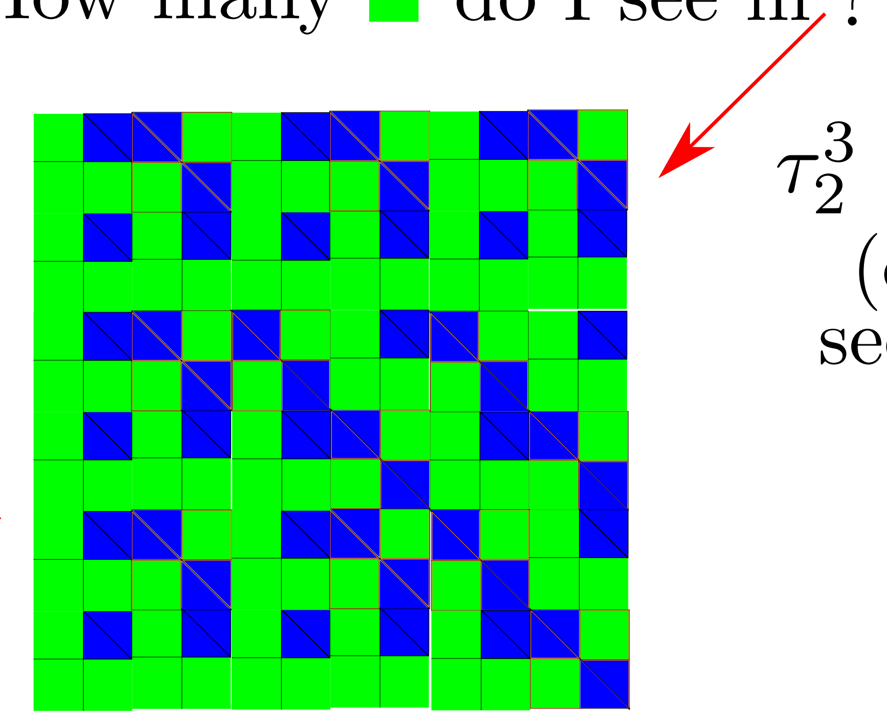
$\tau_2^3 \in \mathrm{Tr}(\mathcal{M}_3) = \mathbb{C}^3$

Bratteli diagram!





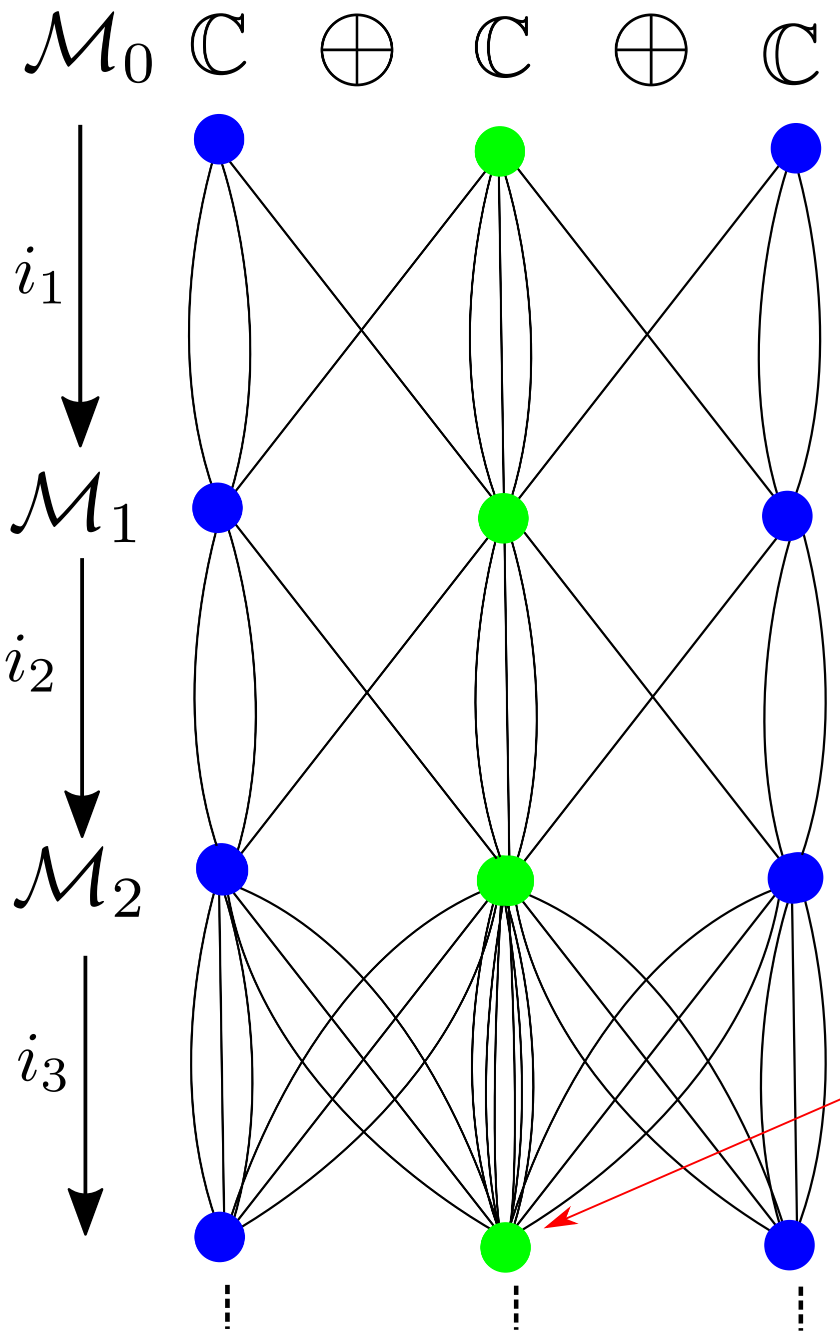
How many do I see in ?

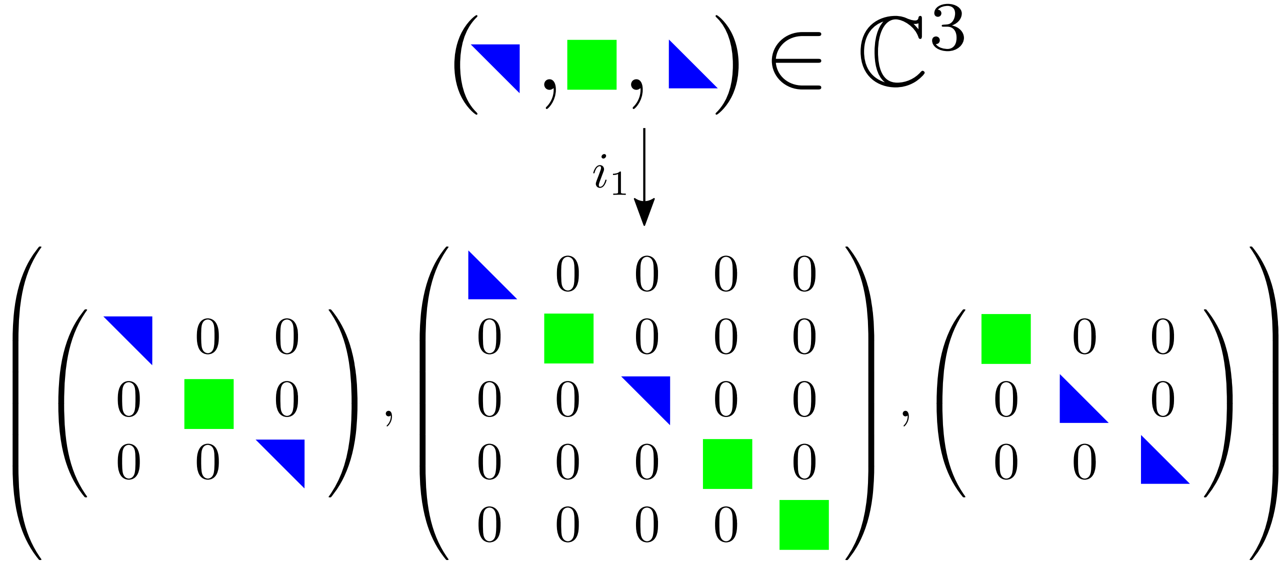


$\tau_2^3 \in \mathrm{Tr}(\mathcal{M}_3) = \mathbb{C}^3$

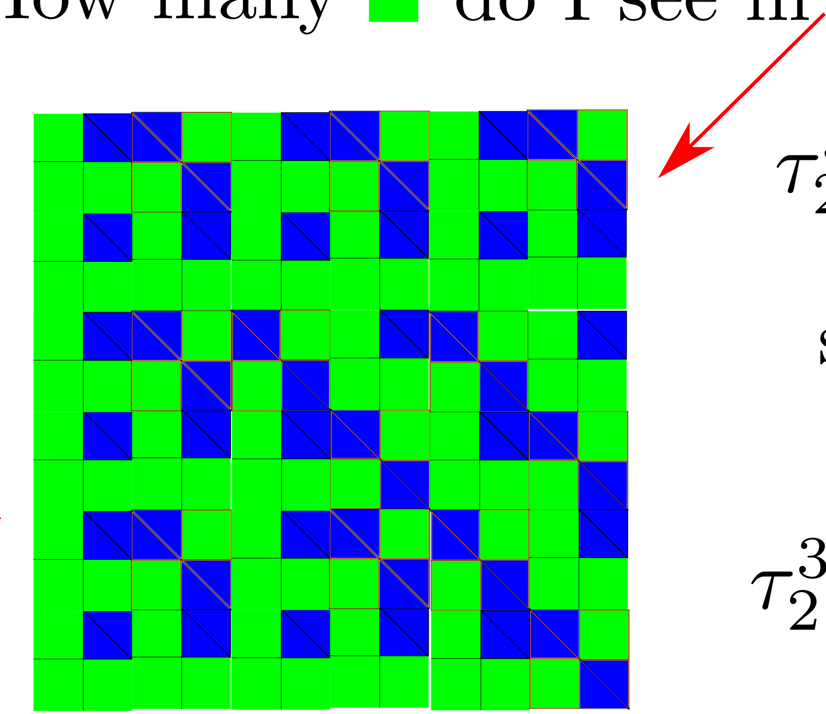
(canonical trace on second summand M_{192})

Bratteli diagram!





How many do I see in ?



$$\tau_2^3 \in \mathrm{Tr}(\mathcal{M}_3) = \mathbb{C}^3$$

(canonical trace on second summand M_{192})

$$\tau_2^3(i_3 \circ i_2 \circ i_1((0,1,0))) = i_1^* i_2^* i_3^* \tau_2^3((0,1,0))$$

Let's do the twist

$$\int e^{-2\pi i \langle \lambda, t \rangle} g_i \circ \varphi_t(\mathcal{T})\, dt$$

Let's do the twist $f_i = \sum \delta_p * h_i$

where p is in the center of mass of tiles of type i

$$\int e^{-2\pi i \langle \lambda, t \rangle} g_i \circ \varphi_t(\mathcal{T})\, dt$$

Let's do the twist

$$f_i = \sum_p \delta_p * h_i$$

where p is in the center of mass of tiles of type i

h_i has compact support

f_i is supported in the union of tiles of type i

$$\int e^{-2\pi i\langle\lambda,t\rangle} g_i \circ \varphi_t(\mathcal{T})\, dt$$

Let's do the twist

$$f_i = \sum_p \delta_p * h_i = g_i \circ \varphi_t(\mathcal{T})$$

for some $g_i : \Omega_x \to \mathbb{R}$

where p is in the center of mass of tiles of type i

h_i has compact support

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$$\int e^{-2\pi i \langle \lambda, t \rangle} g_i \circ \varphi_t(\mathcal{T})\, dt$$

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h_i has compact support

f_i is supported in the union of tiles of type i

For a vertex v, let $\mathcal{P}(v)$ be the associated patch,

$$\int_{\mathcal{P}(v)} e^{-2\pi i \langle \lambda, t \rangle} g_i \circ \varphi_t(\mathcal{T})\, dt$$

Let's do the twist

$$f_i = \sum_p \delta_p * h_i = g_i \circ \varphi_t(\mathcal{T})$$

where p is in the center of mass of tiles of type i

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h_i has compact support

f_i is supported in the union of tiles of type i

For a vertex v, let $\mathcal{P}(v)$ be the associated patch, τ_v the associated trace,

$$\int_{\mathcal{P}(v)} e^{-2\pi i \langle \lambda, t \rangle} g_i \circ \varphi_t(\mathcal{T})\, dt =$$

LET'S DO THE TWIST

$$f_i = \sum_p \delta_p * h_i = g_i \circ \varphi_t(\mathcal{T})$$

for some $g_i : \Omega_x \to \mathbb{R}$

where p is in the center of mass of tiles of type i

h_i has compact support

f_i is supported in the union of tiles of type i

For a vertex v, let $\mathcal{P}(v)$ be the associated patch, τ_v the associated trace, $p_v \in \mathcal{P}(v)$ some point

$$\int_{\mathcal{P}(v)} e^{-2\pi i \langle \lambda, t \rangle} g_i \circ \varphi_t(\mathcal{T})\, dt =$$

Let's do the twist

$$f_i = \sum_p \delta_p * h_i = g_i \circ \varphi_t(\mathcal{T})$$

where p is in the center of mass of tiles of type i

for some $g_i : \Omega_x \to \mathbb{R}$

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For a vertex v, let $\mathcal{P}(v)$ be the associated patch, τ_v the associated trace, $p_v \in \mathcal{P}(v)$ some point

$$\int_{\mathcal{P}(v)} e^{-2\pi i\langle\lambda,t\rangle} g_i \circ \varphi_t(\mathcal{T})\,dt = \int_{\substack{\text{tiles in } \mathcal{P}(v) \\ \text{of type } i}} e^{-2\pi i\langle\lambda,t\rangle} g_i \circ \varphi_t(\mathcal{T})\,dt =$$

Let's do the twist

$$f_i = \sum_p \delta_p * h_i = g_i \circ \varphi_t(\mathcal{T})$$

for some $g_i : \Omega_x \to \mathbb{R}$

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For a vertex v, let $\mathcal{P}(v)$ be the associated patch, τ_v the associated trace, $p_v \in \mathcal{P}(v)$ some point

$$\int_{\mathcal{P}(v)} e^{-2\pi i \langle \lambda, t\rangle} g_i \circ \varphi_t(\mathcal{T})\, dt = \int_{\substack{\text{tiles in } \mathcal{P}(v) \\ \text{of type } i}} e^{-2\pi i \langle \lambda, t\rangle} g_i \circ \varphi_t(\mathcal{T})\, dt =$$

$$\sum_{\ell=1}^{\tau_v(e_i)} \int_{t_i} e^{-2\pi i \langle \lambda, t - \tau_\ell\rangle} h_i(t)\, dt$$

Let's do the twist $f_i = \sum_p \delta_p * h_i = g_i \circ \varphi_t(\mathcal{T})$

for some $g_i : \Omega_x \to \mathbb{R}$

where p is in the center of mass of tiles of type i

h_i has compact support

f_i is supported in the union of tiles of type i

For a vertex v, let $\mathcal{P}(v)$ be the associated patch, τ_v the associated trace, $p_v \in \mathcal{P}(v)$ some point

$$\int_{\mathcal{P}(v)} e^{-2\pi i \langle \lambda, t \rangle} g_i \circ \varphi_t(\mathcal{T})\, dt = \int_{\substack{\text{tiles in } \mathcal{P}(v) \\ \text{of type } i}} e^{-2\pi i \langle \lambda, t \rangle} g_i \circ \varphi_t(\mathcal{T})\, dt =$$

$$\sum_{\ell=1}^{\tau_v(e_i)} \int_{t_i} e^{-2\pi i \langle \lambda, t - \tau_\ell \rangle} h_i(t)\, dt = \sum_{\ell=1}^{\tau_v(e_i)} e^{2\pi i \langle \lambda, \tau_\ell \rangle} \int_{t_i} e^{-2\pi i \langle \lambda, t \rangle} h_i(t)\, dt$$

Let's do the twist

$$f_i = \sum_p \delta_p * h_i = g_i \circ \varphi_t(\mathcal{T})$$

where p is in the center of mass of tiles of type i

for some $g_i : \Omega_x \to \mathbb{R}$

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f_i is supported in the union of tiles of type i

For a vertex v, let $\mathcal{P}(v)$ be the associated patch, τ_v the associated trace, $p_v \in \mathcal{P}(v)$ some point

$$\int_{\mathcal{P}(v)} e^{-2\pi i\langle\lambda,t\rangle} g_i \circ \varphi_t(\mathcal{T})\,dt = \int_{\substack{\text{tiles in } \mathcal{P}(v)\\ \text{of type } i}} e^{-2\pi i\langle\lambda,t\rangle} g_i \circ \varphi_t(\mathcal{T})\,dt =$$

$$\sum_{\ell=1}^{\tau_v(e_i)} \int_{t_i} e^{-2\pi i\langle\lambda,t-\tau_\ell\rangle} h_i(t)\,dt = \sum_{\ell=1}^{\tau_v(e_i)} e^{2\pi i\langle\lambda,\tau_\ell\rangle} \int_{t_i} e^{-2\pi i\langle\lambda,t\rangle} h_i(t)\,dt$$

$$= \sum_{\ell=1}^{\tau_v(e_i)} e^{2\pi i\langle\lambda,\tau_\ell\rangle} \hat{h}_i(\lambda)$$

Let's do the twist

$$f_i = \sum_p \delta_p * h_i = g_i \circ \varphi_t(\mathcal{T})$$

where p is in the center of mass of tiles of type i

for some $g_i : \Omega_x \to \mathbb{R}$

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For a vertex v, let $\mathcal{P}(v)$ be the associated patch, τ_v the associated trace, $p_v \in \mathcal{P}(v)$ some point

$$\int_{\mathcal{P}(v)} e^{-2\pi i\langle\lambda,t\rangle} g_i \circ \varphi_t(\mathcal{T})\,dt = \int_{\substack{\text{tiles in } \mathcal{P}(v) \\ \text{of type } i}} e^{-2\pi i\langle\lambda,t\rangle} g_i \circ \varphi_t(\mathcal{T})\,dt =$$

$$\sum_{\ell=1}^{\tau_v(e_i)} \int_{t_i} e^{-2\pi i\langle\lambda,t-\tau_\ell\rangle} h_i(t)\,dt = \sum_{\ell=1}^{\tau_v(e_i)} e^{2\pi i\langle\lambda,\tau_\ell\rangle} \int_{t_i} e^{-2\pi i\langle\lambda,t\rangle} h_i(t)\,dt$$

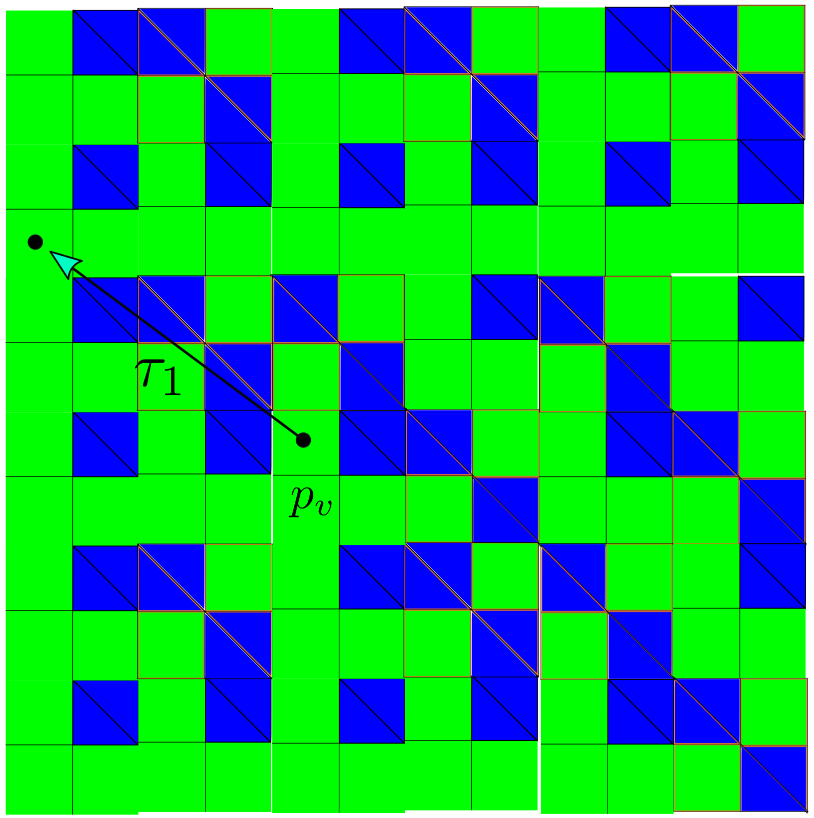
$$= \sum_{\ell=1}^{\tau_v(e_i)} e^{2\pi i\langle\lambda,\tau_\ell\rangle} \hat{h}_i(\lambda) = \hat{h}_i(\lambda) \sum_{\ell=1}^{\tau_v(e_i)} e^{2\pi i\langle\lambda,\tau_\ell\rangle}$$

Let's do the twist

$$\left| \int_{\mathcal{P}(v)} e^{-2\pi i \langle \lambda, t \rangle} g_i \circ \varphi_t(\mathcal{T}) \, dt \right| \leq |\hat{h}_i(\lambda)| \left| \sum_{\ell=1}^{\tau_v(e_i)} e^{2\pi i \langle \lambda, \tau_\ell \rangle} \right|$$

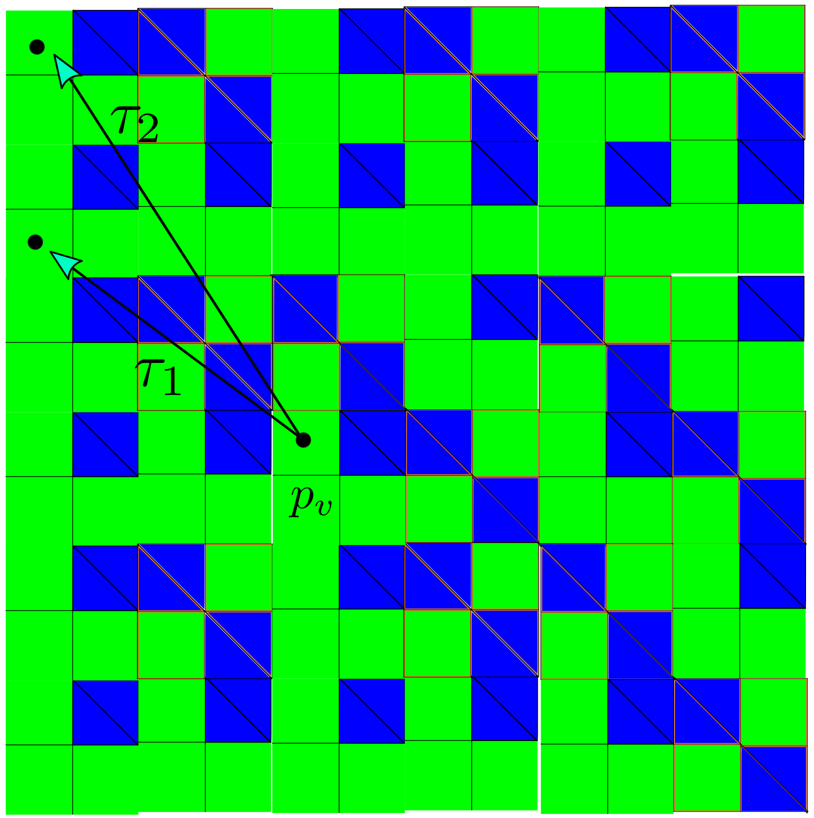
Let's do the twist

$$\left|\int_{\mathcal{P}(v)} e^{-2\pi i\langle \lambda, t\rangle} g_i \circ \varphi_t(\mathcal{T})\, dt\right| \leq |\hat{h}_i(\lambda)| \left|\sum_{\ell=1}^{\tau_v(e_i)} e^{2\pi i\langle \lambda, \tau_\ell\rangle}\right|$$



Let's do the twist

$$\left|\int_{\mathcal{P}(v)} e^{-2\pi i\langle\lambda,t\rangle} g_i\circ\varphi_t(\mathcal{T})\,dt\right| \le |\hat{h}_i(\lambda)| \left|\sum_{\ell=1}^{\tau_v(e_i)} e^{2\pi i\langle\lambda,\tau_\ell\rangle}\right|$$



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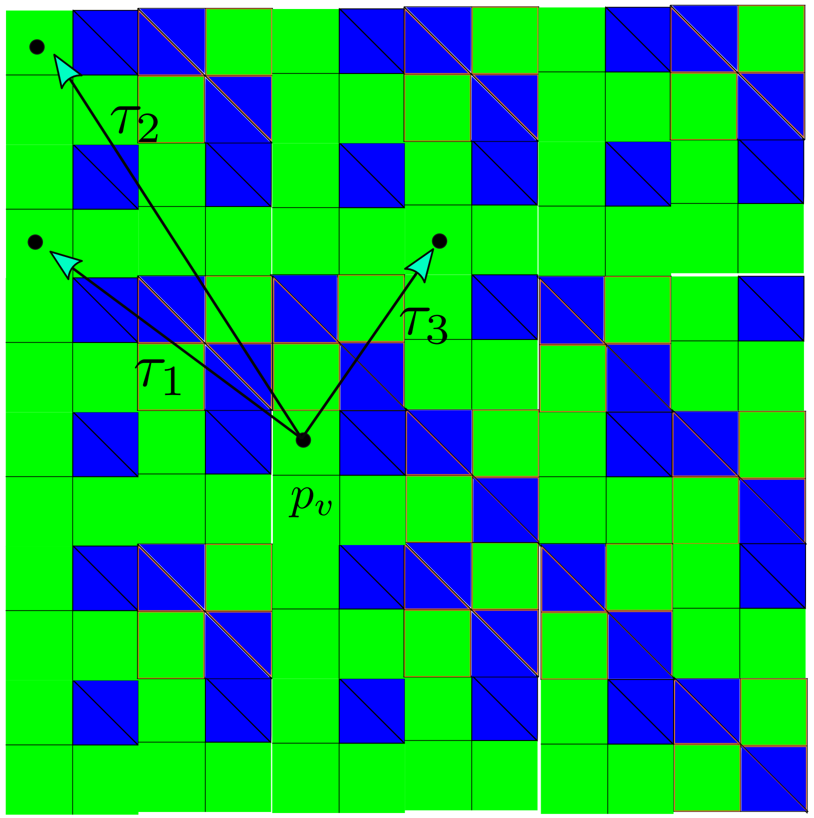
$$\left| \int_{\mathcal{P}(v)} e^{-2\pi i \langle \lambda, t \rangle} g_i \circ \varphi_t(\mathcal{T}) \, dt \right| \leq |\hat{h}_i(\lambda)| \left| \sum_{\ell=1}^{\tau_v(e_i)} e^{2\pi i \langle \lambda, \tau_\ell \rangle} \right|$$



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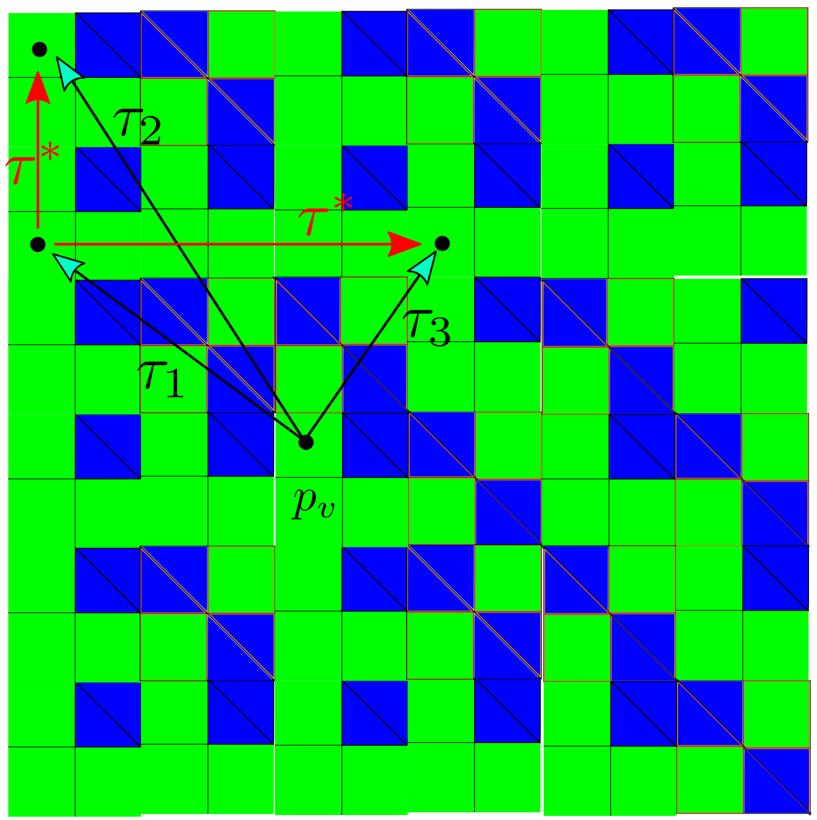
$$\left|\int_{\mathcal{P}(v)} e^{-2\pi i\langle\lambda,t\rangle} g_i\circ\varphi_t(\mathcal{T})\,dt\right| \le |\hat{h}_i(\lambda)| \left|\sum_{\ell=1}^{\tau_v(e_i)} e^{2\pi i\langle\lambda,\tau_\ell\rangle}\right|$$

τ^* is a return vector between tiles of type i



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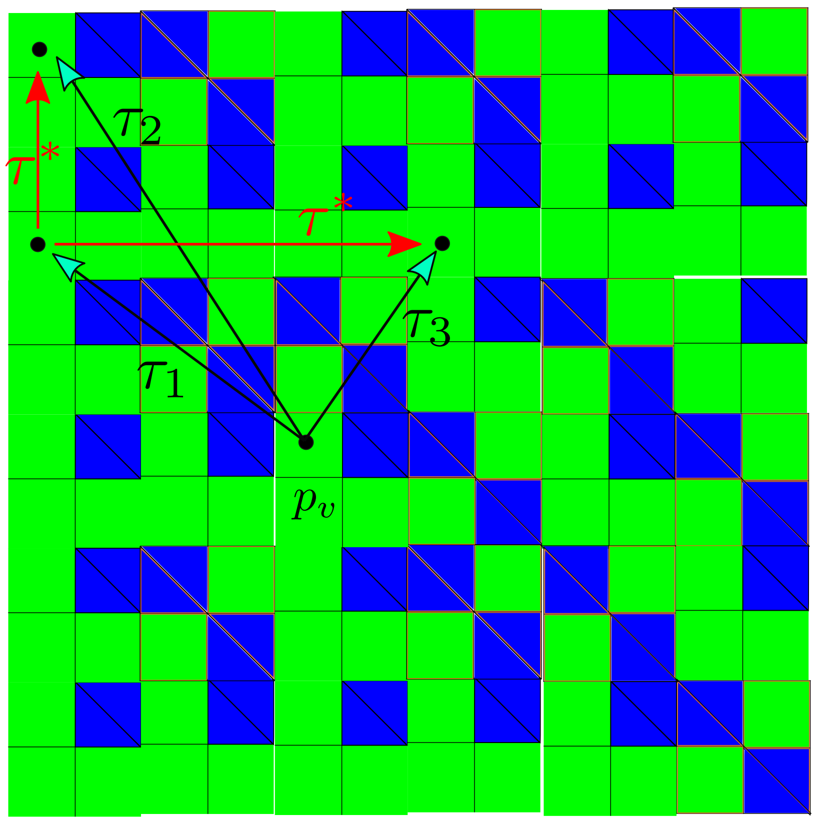
$$\left|\int_{\mathcal{P}(v)} e^{-2\pi i\langle\lambda,t\rangle} g_i\circ\varphi_t(\mathcal{T})\,dt\right| \leq |\hat{h}_i(\lambda)| \left|\sum_{\ell=1}^{\tau_v(e_i)} e^{2\pi i\langle\lambda,\tau_\ell\rangle}\right|$$

τ^* is a return vector between tiles of type i

$$\leq |\hat{h}_i(\lambda)| \left|\tau_v(e_i) - 2 + \left|1 + e^{2\pi i\langle\lambda,\tau^*\rangle}\right|\right|$$



Let's do the twist

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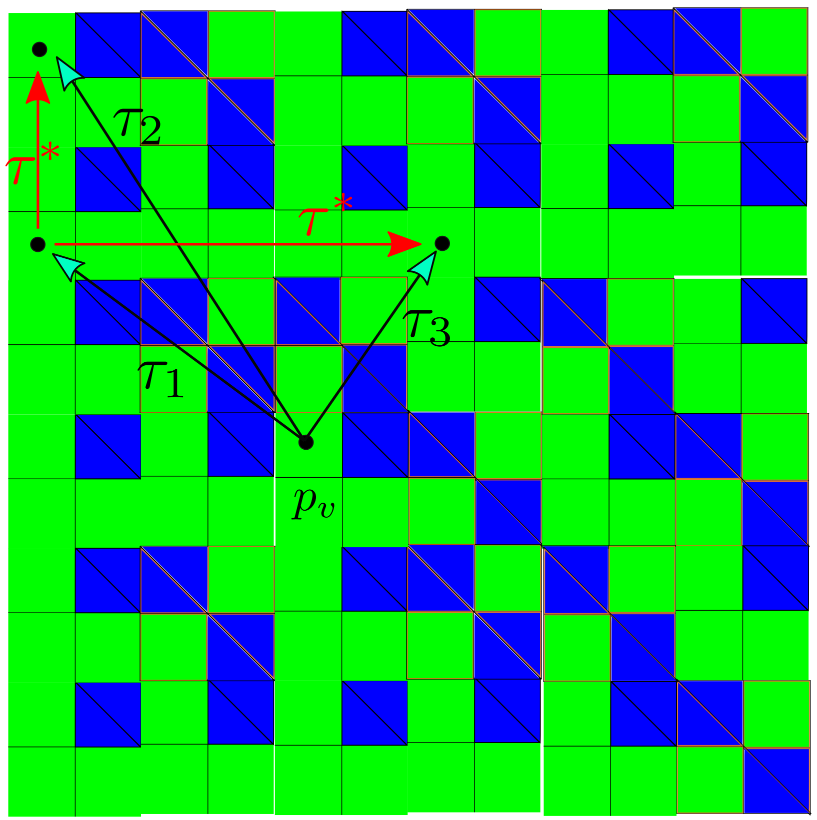
For **any** return vector τ^* in $\mathcal{P}(v)$



Let's do the twist

$$\left|\int_{\mathcal{P}(v)} e^{-2\pi i\langle\lambda,t\rangle} g_i\circ\varphi_t(\mathcal{T})\,dt\right| \leq |\hat{h}_i(\lambda)| \left|\sum_{\ell=1}^{\tau_v(e_i)} e^{2\pi i\langle\lambda,\tau_\ell\rangle}\right|$$

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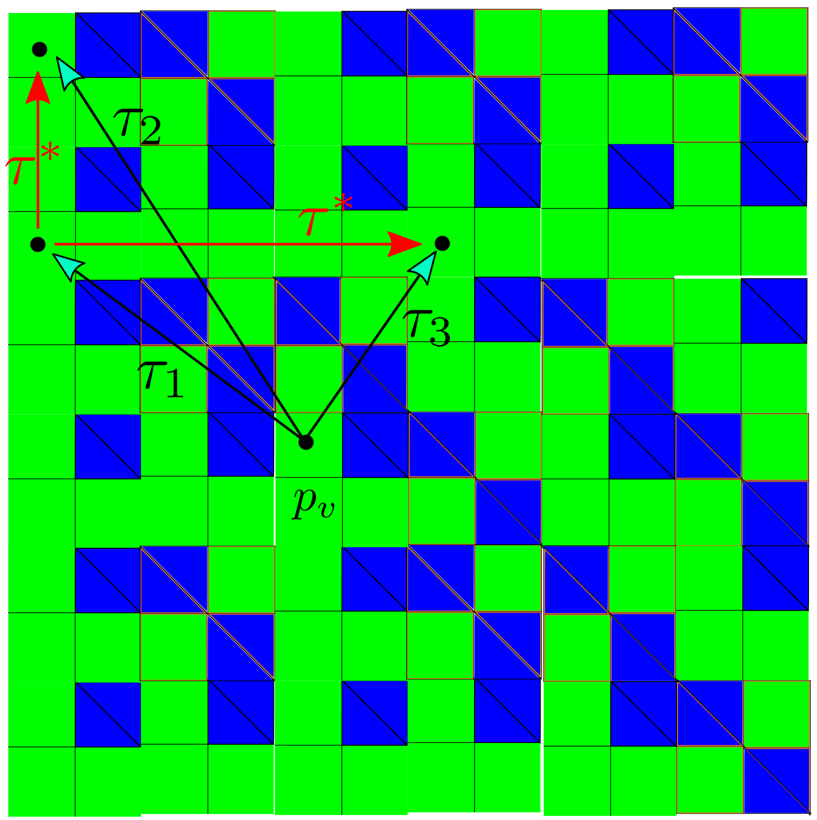
For **any** return vector τ^* in $\mathcal{P}(v)$

$$\leq |\hat{h}_i(\lambda)|(\tau_v(e_i) - \frac{1}{2}\|\langle\lambda,\tau^*\rangle\|^2_{\mathbb{R}/\mathbb{Z}})$$



Let's do the twist

$$\left|\int_{\mathcal{P}(v)} e^{-2\pi i\langle\lambda,t\rangle} g_i\circ\varphi_t(\mathcal{T})\,dt\right| \le |\hat{h}_i(\lambda)| \left|\sum_{\ell=1}^{\tau_v(e_i)} e^{2\pi i\langle\lambda,\tau_\ell\rangle}\right|$$

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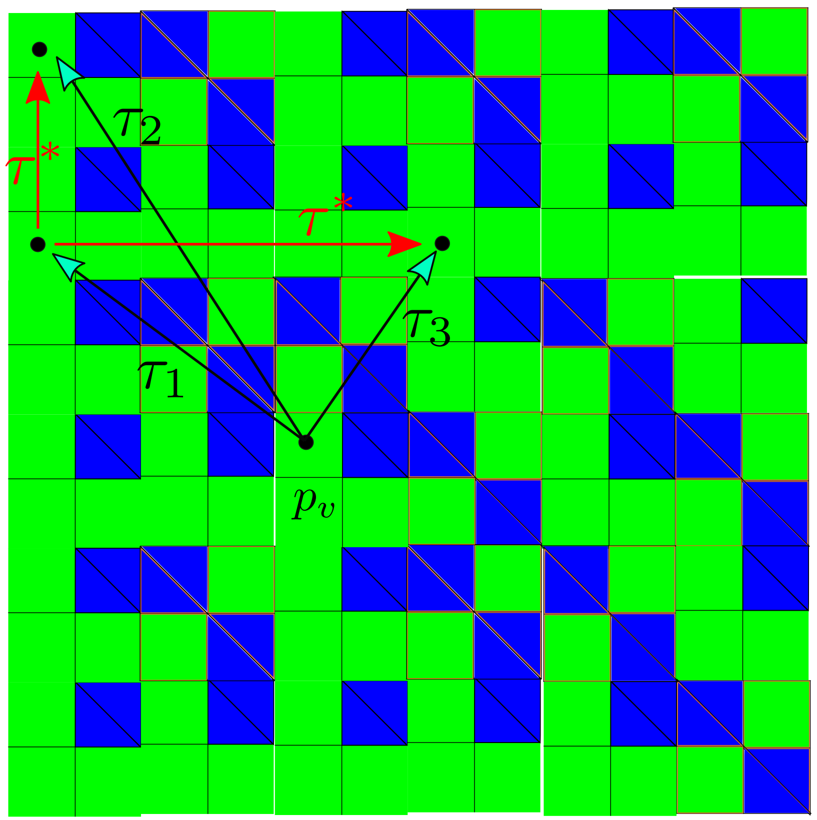
(Given that $|1+e^{2\pi i\omega}| \le 2 - \frac{1}{2}\|\omega\|^2_{\mathbb{R}/\mathbb{Z}}$)
(Bufetov-Solomyak)

For **any** return vector τ^* in $\mathcal{P}(v)$

$$\le |\hat{h}_i(\lambda)|\left(\tau_v(e_i) - \frac{1}{2}\|\langle\lambda,\tau^*\rangle\|^2_{\mathbb{R}/\mathbb{Z}}\right)$$



Let's do the twist

$$\left|\int_{\mathcal{P}(v)} e^{-2\pi i\langle\lambda,t\rangle} g_i\circ\varphi_t(\mathcal{T})\,dt\right| \leq |\hat{h}_i(\lambda)|\left|\sum_{\ell=1}^{\tau_v(e_i)} e^{2\pi i\langle\lambda,\tau_\ell\rangle}\right|$$

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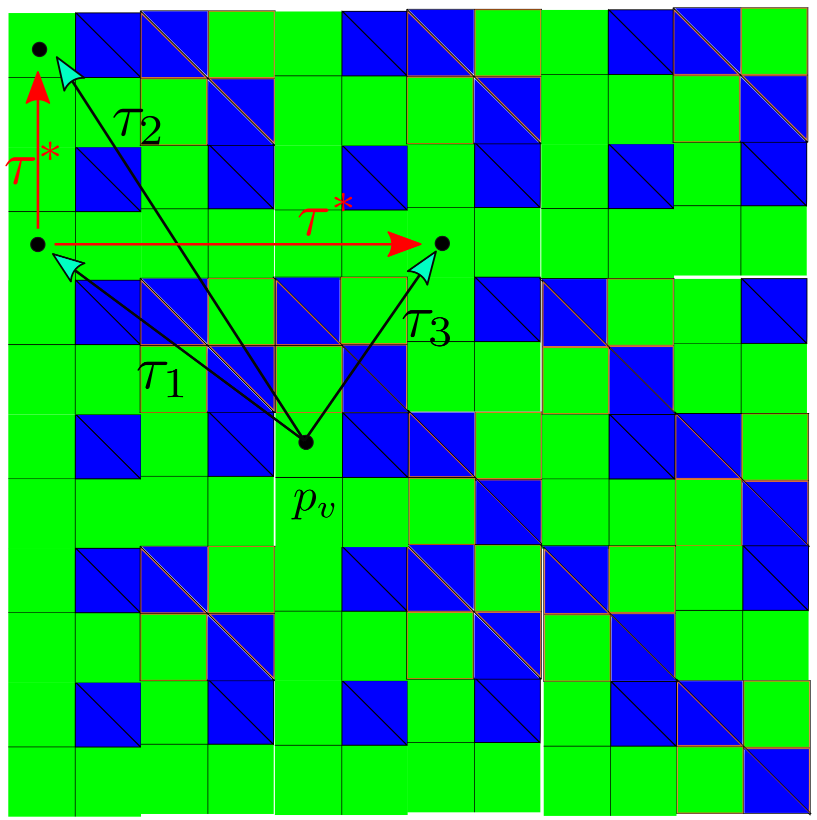
growth controlled by the trace cocycle (cocycle on $H^d(\Omega_x;\mathbb{R})$)



Let's do the twist

$$\left|\int_{\mathcal{P}(v)} e^{-2\pi i\langle\lambda,t\rangle} g_i\circ\varphi_t(\mathcal{T})\,dt\right| \leq |\hat{h}_i(\lambda)|\left|\sum_{\ell=1}^{\tau_v(e_i)} e^{2\pi i\langle\lambda,\tau_\ell\rangle}\right|$$

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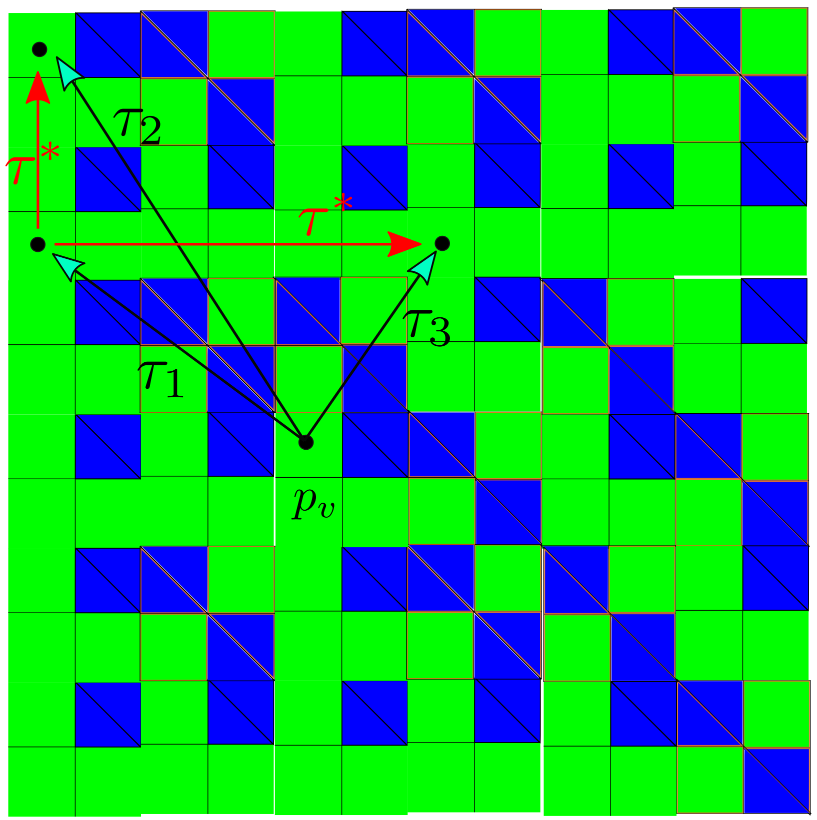
$$\leq |\hat{h}_i(\lambda)|\left(\tau_v(e_i) - \frac{1}{2}\|\langle\lambda,\tau^*\rangle\|^2_{\mathbb{R}/\mathbb{Z}}\right)$$

growth controlled by the trace cycle (cocycle on $H^d(\Omega_x;\mathbb{R})$)

growth controlled by the return vector cocycle (cocycle on $H^1(\Omega_x;\mathbb{R})$)



Let's do the twist

$$\left|\int_{\mathcal{P}(v)} e^{-2\pi i\langle\lambda,t\rangle} g_i\circ\varphi_t(\mathcal{T})\,dt\right| \le |\hat{h}_i(\lambda)|\left|\sum_{\ell=1}^{\tau_v(e_i)} e^{2\pi i\langle\lambda,\tau_\ell\rangle}\right|$$

τ^* is a return vector between tiles of type i

$$\le |\hat{h}_i(\lambda)|\left|\tau_v(e_i)-2+\left|1+e^{2\pi i\langle\lambda,\tau^*\rangle}\right|\right|$$

(Given that $|1+e^{2\pi i\omega}|\le 2-\frac{1}{2}\|\omega\|^2_{\mathbb{R}/\mathbb{Z}}$)
(Bufetov-Solomyak)

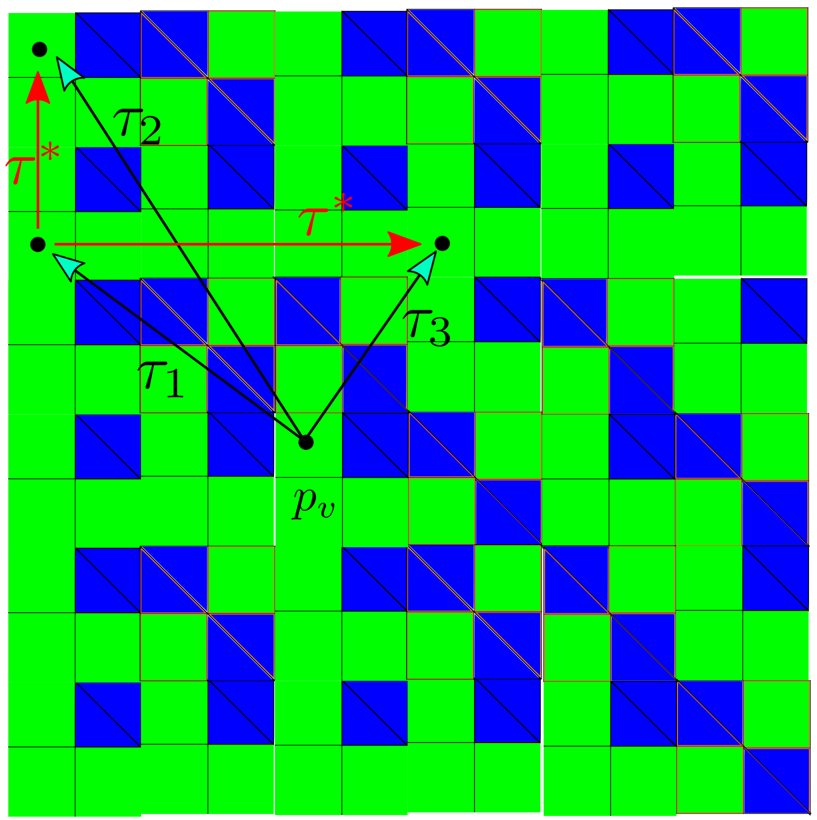
For **any** return vector τ^* in $\mathcal{P}(v)$

$$\le |\hat{h}_i(\lambda)|\left(\tau_v(e_i)-\frac{1}{2}\|\langle\lambda,\tau^*\rangle\|^2_{\mathbb{R}/\mathbb{Z}}\right)$$

growth controlled by the trace cocycle (cocycle on $H^d(\Omega_x;\mathbb{R})$)

growth controlled by the return vector cocycle (cocycle on $H^1(\Omega_x;\mathbb{R})$)

When $d=1$ there is one cocycle to care about.



Let's do the twist

$$\left|\int_{\mathcal{P}(v)} e^{-2\pi i\langle\lambda,t\rangle} g_i\circ\varphi_t(\mathcal{T})\,dt\right| \leq |\hat{h}_i(\lambda)| \left|\sum_{\ell=1}^{\tau_v(e_i)} e^{2\pi i\langle\lambda,\tau_\ell\rangle}\right|$$

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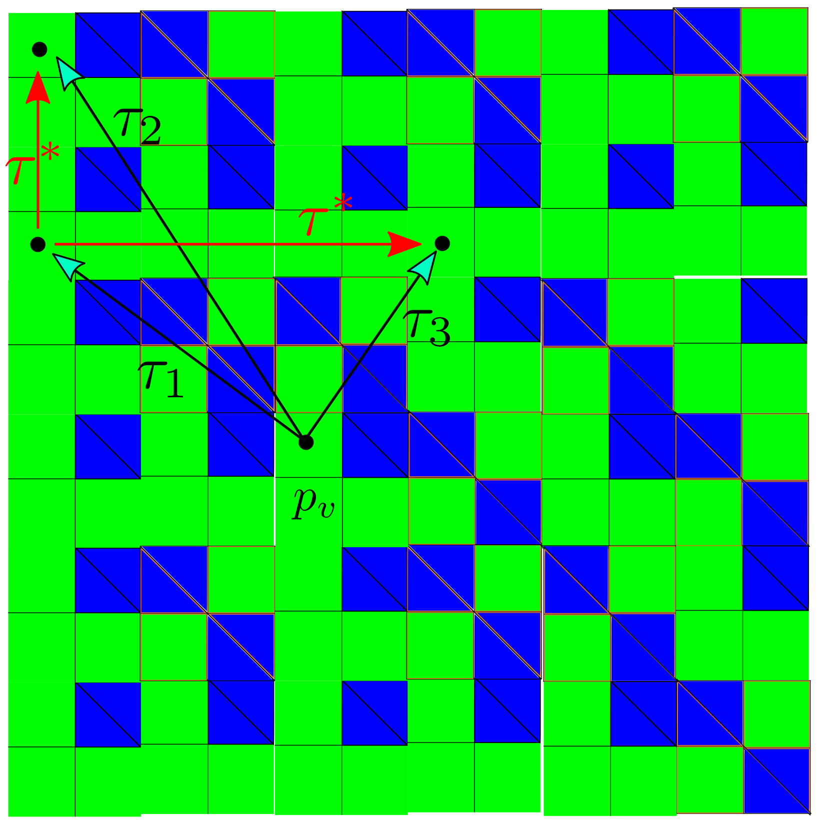

growth controlled by the trace cocycle (cocycle on $H^d(\Omega_x;\mathbb{R})$)

growth controlled by the return vector cocycle (cocycle on $H^1(\Omega_x;\mathbb{R})$)

When $d=1$ there is one cocycle to care about.
When $d>1$ these are different!

Let's do the twist

$$\left|\int_{\mathcal{P}(v)} e^{-2\pi i\langle\lambda,t\rangle} g_i\circ\varphi_t(\mathcal{T})\,dt\right| \leq |\hat{h}_i(\lambda)| \left|\sum_{\ell=1}^{\tau_v(e_i)} e^{2\pi i\langle\lambda,\tau_\ell\rangle}\right|$$

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(Given that $|1+e^{2\pi i\omega}| \leq 2 - \frac{1}{2}\|\omega\|^2_{\mathbb{R}/\mathbb{Z}}$)
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For **any** return vector τ^* in $\mathcal{P}(v)$

$$\leq |\hat{h}_i(\lambda)|\left(\tau_v(e_i) - \frac{1}{2}\|\langle\lambda,\tau^*\rangle\|^2_{\mathbb{R}/\mathbb{Z}}\right)$$

growth controlled by the trace cocycle (cocycle on $H^d(\Omega_x;\mathbb{R})$)

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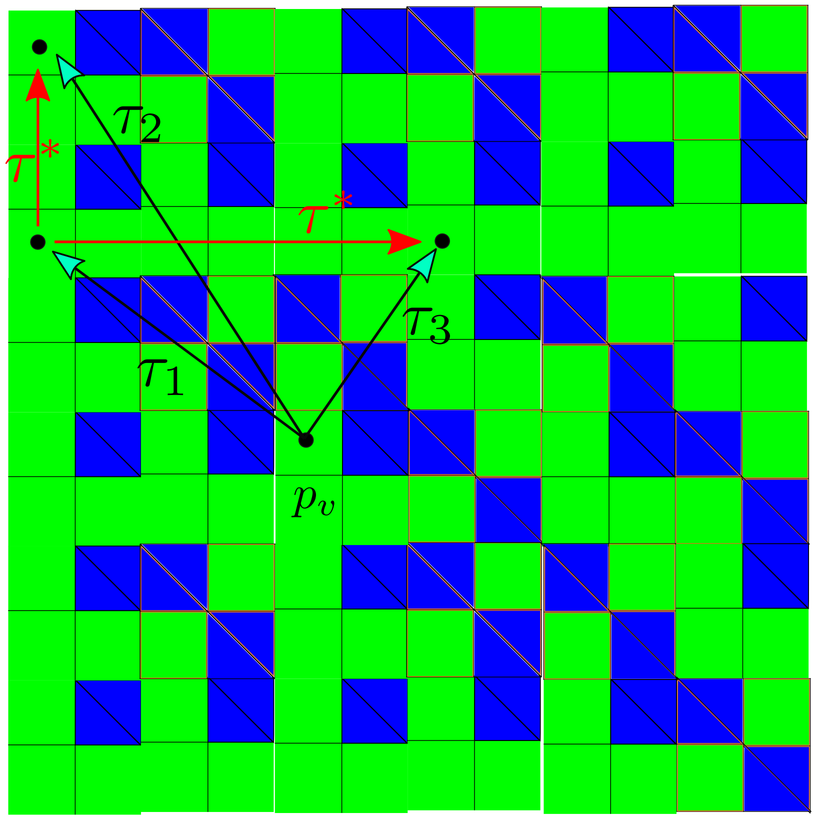

Need to stay away from lattice points under the renormalization dynamics (cocycle dynamics). Use the Erdos-Kahane method (B-S) to estimate the dimension of set of deformation parameters which does not stay away from lattice points. This set has codimension at most $\dim(E^+_x) - d$, which is why if you have $d+1$ positive exponents the bad set is small.