Measure rigidity and orbit closure classification of random walks on surfaces

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Setting

Given a manifold M, a point $x \in M$ and a semigroup Γ acting on M,

what can we say about:

- the orbit of x under Γ,

$$\mathrm{Orbit}(x, \Gamma) := \{\varphi(x) \mid \varphi \in \Gamma\}?$$

- the Γ-invariant probability measures ν on M?

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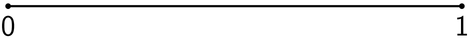
When can we classify all of them?

Circle

Say $M = S^1 = [0,1]/\sim$, $\quad f(x) = 3x \bmod 1$, $\quad \Gamma = \langle f \rangle$ is cyclic,



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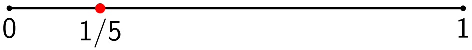

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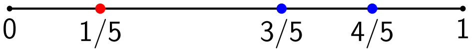

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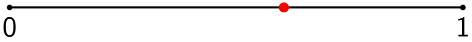

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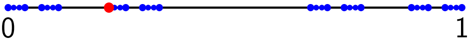

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- By the pointwise ergodic theorem, we know that for almost every point $x \in S^1$, $\mathrm{Orbit}(x,\Gamma)$ is dense (in fact equidistributed w.r.t. Leb).
- But there are points $x \in S^1$ where $\mathrm{Orbit}(x,\Gamma)$ is neither finite nor dense, for instance for certain $x \in S^1$, the closure of its orbit

$$\overline{\mathrm{Orbit}(x,\Gamma)} = \text{ middle third Cantor set.}$$

(And many orbit closures of Hausdorff dimension between 0 and 1!)

Furstenberg's $\times 2 \times 3$ problem

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For invariant measures...

Conjecture (Furstenberg, 1967)

Every ergodic Γ-invariant probability measure ν on S^1 is either finitely supported or the Lebesgue measure.

Free group action on 2-torus

For dim $M = 2$, one observes similar phenomenon. Say $M = \mathbb{T}^2$, and $\Gamma = \langle f, g \rangle$ with

$$f = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \qquad g = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \quad \in SL_2(\mathbb{Z})$$

which acts on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ by left multiplication.

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Stationary measure

In fact, the theorem of BFLM classifies stationary measures on $\mathbb{T}^d$.

Let X be a metric space, G be a group acting continuously on X. Let μ be a probability measure on G.

Definition

A measure ν on X is μ-stationary if

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Previous example: $X = \mathbb{T}^2, G = SL_2(\mathbb{Z}), \Gamma = \langle \mathrm{supp}\ \mu \rangle = \langle A, B \rangle \subset G$,

$$\mu = \frac{1}{2}(\delta_A + \delta_B), \quad \text{where} \quad A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

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- Stationary measures are relevant for equidistribution problems.

Zariski dense toral automorphism

Theorem (Bourgain-Furman-Lindenstrauss-Mozes, Benoist-Quint)

Let μ be a compactly supported probability measure on $SL_d(\mathbb{Z})$.
If $\Gamma = \langle \mathrm{supp}\ \mu \rangle$ is a Zariski dense subsemigroup of $SL_d(\mathbb{R})$, then

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- The Zariski density assumption is necessary since the theorem is false for say cyclic Γ generated by a hyperbolic element in $SL_d(\mathbb{Z})$.
- The second conclusion implies that under the given assumptions, every μ-stationary measure is Γ-invariant (i.e. stiffness).

Homogeneous Setting

The theorem of Benoist-Quint works more generally for homogeneous spaces G/Λ.

Theorem (Benoist-Quint, 2011)

Let G be a connected simple real Lie group, Λ be a lattice in G, μ be a compactly supported probability measure on G. If $\Gamma = \langle \mathrm{supp}\ \mu \rangle$ is a Zariski dense subsemigroup of G, then

- *For all $x \in G/\Lambda$, $\mathrm{Orbit}(x, \Gamma)$ is either finite or dense.*
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More general results in the homogeneous setting by Benoist-Quint (semisimple setting), Eskin-Lindenstrauss (uniform expansion on $\mathfrak{g}$) etc.

Non-homogeneous setting

Let M be a closed manifold with (normalized) volume measure vol,
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Under what condition on μ and/or Γ do we have that

- For all $x \in M$, $\mathrm{Orbit}(x, \Gamma)$ is either finite or dense.
- Every ergodic μ-stationary probability measure ν on M is either finitely supported or vol.
- Every infinite orbit equidistributes on M?

Uniform expansion

Definition

Let M be a Riemannian manifold, μ be a probability measure on $\mathrm{Diff}^2_{\mathrm{vol}}(M)$. We say that μ is uniformly expanding if there exists $C > 0$ and $N \in \mathbb{N}$ such that for all $x \in M$ and nonzero $v \in T_x M$,

$$\int_{\mathrm{Diff}^2_{\mathrm{vol}}(M)} \log \frac{\|D_x f(v)\|}{\|v\|} d\mu^{(N)}(f) > C > 0.$$

Here $\mu^{(N)} := \mu * \mu * \cdots * \mu$ is the N-th convolution power of μ.

In other words, the random walk w.r.t. μ expands every vector $v \in T_x M$ at every point $x \in M$ on average (might be contracted by a specific word though!)

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Remark: Uniform expansion is an open condition, expected to be generic.

Main result

Theorem (C.)

Let M be a closed 2-manifold with volume measure vol. Let μ be a compactly supported probability measure on $\mathrm{Diff}^2_{\mathrm{vol}}(M)$ that is uniformly expanding, and $\Gamma := \langle \mathrm{supp}\ \mu \rangle$. Then

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Remark

- *For $M = \mathbb{T}^2$ and μ supported on $SL_2(\mathbb{Z})$, if $\Gamma = \langle \mathrm{supp}\ \mu \rangle$ is Zariski dense in $SL_2(\mathbb{R})$, then μ is uniformly expanding.*

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- *Since uniform expansion is an open condition, so the conclusion holds for small perturbations of Zariski dense toral automorphisms in $\mathrm{Diff}^2_{\mathrm{vol}}(M)$ too.*

Verify uniform expansion

How hard is it to verify the uniform expansion condition? We checked it in two settings:

1. Discrete perturbation of the standard map (verified by hand)
2. $\mathrm{Out}(F_2)$-action on the character variety $\mathsf{Hom}(F_2, \mathrm{SU}(2)) /\!\!/ \mathrm{SU}(2)$ (verified numerically).

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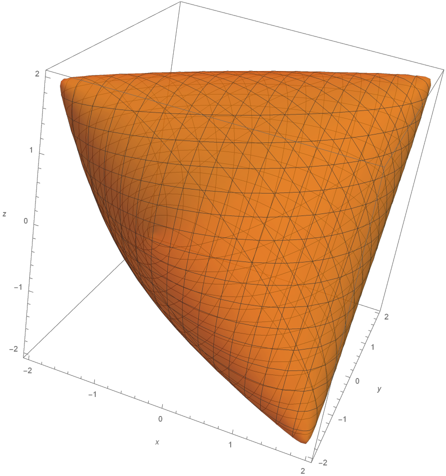
Application: Out(F2)-action on character variety

The character variety $\text{Hom}(F_2, \mathrm{SU}(2)) /\!\!/ \mathrm{SU}(2)$ can be embedded in $\mathbb{R}^3$ via trace coordinates, with image given by

$$\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 - xyz - 2 \in [-2, 2]\} \subset \mathbb{R}^3.$$



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Moreover, under the natural action of $\mathrm{Out}(F_2)$, the ergodic components are the compact surfaces

$$\{x^2 + y^2 + z^2 - xyz - 2 = k\} \subset \mathbb{R}^3$$

for $k \in [-2, 2]$, corresponding to relative character varieties $\mathrm{Hom}_k(F_2, \mathrm{SU}(2)) /\!\!/ \mathrm{SU}(2)$. Under such identification, the action of $\mathrm{Out}(F_2)$ is generated by two Dehn twists

$$T_X \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ z \\ xz - y \end{pmatrix}, \qquad T_Y \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ y \\ yz - x \end{pmatrix}.$$

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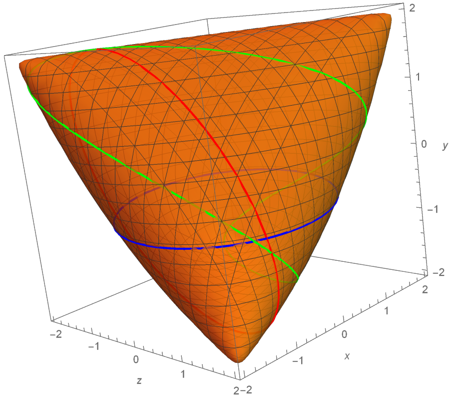
For $k = 1.99$, the relative character variety is

$$\{x^2 + y^2 + z^2 - xyz - 2 = k\} \subset \mathbb{R}^3$$

with maps

$$T_X(x, y, z) = (x, z, xz - y), \qquad T_Y(x, y, z) = (z, y, yz - x).$$



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$$\int_{\mathrm{Diff}^2(M)} \log \frac{\|D_P f(v)\|}{\|v\|} d\mu^{(N)}(f) > C > 0.$$

Given the explicit form of both the compact surface and the maps, one can verify uniform expansion numerically:

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Time complexity: $O(\lambda^6 A^2)$, where λ, A are C^1 and C^2 norms of f.

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Theorem (C.)

For k near 2, $\mu = \frac{1}{2}\delta_{T_X} + \frac{1}{2}\delta_{T_Y}$ is uniformly expanding on $Hom_k(F_2, \mathrm{SU}(2)) /\!\!/ \mathrm{SU}(2)$.

Corollary

For k near 2, let $X = Hom_k(F_2, \mathrm{SU}(2)) /\!\!/ \mathrm{SU}(2)$, then

- *every $\mathrm{Out}(F_2)$-orbit on X is either finite or dense.*
- *Every infinite orbit equidistribute on X.*
- *Every ergodic $\mathrm{Out}(F_2)$-invariant measure on X is either finitely supported or the natural volume measure.*

Application: Out(F2)-action on character variety

Remark:

1. The topological statement was obtained by Previte and Xia for all $k \in [-2, 2]$ with a completely different method, using crucially the fact that $\mathrm{Out}(F_2)$ is generated by Dehn twists.

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2. Our method is readily applicable for proper subgroups Γ of $\mathrm{Out}(F_2)$, including those without any powers of Dehn twists. It is only limited by computational power.
3. Are there faster algorithms to verify uniform expansion? Likely.

Application: Out(F2)-action on character variety

Remark:

1. The topological statement was obtained by Previte and Xia for all $k \in [-2, 2]$ with a completely different method, using crucially the fact that $\mathrm{Out}(F_2)$ is generated by Dehn twists.
2. Our method is readily applicable for proper subgroups Γ of $\mathrm{Out}(F_2)$, including those without any powers of Dehn twists. It is only limited by computational power.
3. Are there faster algorithms to verify uniform expansion? Likely.

Thank you!

Part II

Proof of main statement

Proof using result of Brown and Rodriguez Hertz

Theorem (Brown-Rodriguez Hertz, 2017)

Let M be a closed 2-manifold. Let μ be a measure on $\mathrm{Diff}^2_{\mathrm{vol}}(M)$, and $\Gamma := \langle \mathrm{supp}\ \mu \rangle$. Let ν be an ergodic hyperbolic μ-stationary measure on M. Then at least one of the following three possibilities holds:

1. *ν is finitely supported.*

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3. *For ν-a.e. $x \in M$, there exists $v \in \mathbb{P}(T_x M)$ that is contracted by $\mu^{\mathbb{N}}$-almost every word ω ("Stable distribution is non-random" in ν).*

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1. Uniform expansion (UE) implies hyperbolicity and rules out (3).
2. UE and some version of the Hopf argument (related to ideas of Dolgopyat-Krikorian) show that $\nu = \mathrm{vol}$ in (2) (global ergodicity).
3. UE together with techniques (Margulis function) originated from Eskin-Margulis show that the classification of stationary measures implies equidistribution and orbit closure classification.

Result of Brown and Rodriguez Hertz

Thus uniform expansion is stronger than the assumptions of Brown-Rodriguez Hertz. But in some sense this is best possible.

Proposition (C.)

Let M be a closed 2-manifold. Let μ be a measure on $\mathrm{Diff}^2_{\mathrm{vol}}(M)$. Then μ is uniformly expanding if and only if for every ergodic μ-stationary measure ν on M,

1. *ν is hyperbolic,*
2. *Stable distribution is not non-random in ν.*

Application: Perturbation of standard map

Application: Perturbation of standard map

The Chirikov standard map is a map on $\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$ given by

$$\Phi_L(I,\theta) = (I + L\sin\theta, \theta + I + L\sin\theta)$$

for a parameter $L > 0$. Under the coordinate change $x = \theta, \quad y = \theta - I$, Φ_L conjugates to (by abuse of notation)

$$\Phi_L(x,y) = (L\sin x + 2x - y, x),$$

with differential map

$$D\Phi_L = \begin{pmatrix} L\cos x + 2 & -1 \\ 1 & 0 \end{pmatrix}.$$

Application: Perturbation of standard map

$$D\Phi_L = \begin{pmatrix} L\cos x + 2 & -1 \\ 1 & 0 \end{pmatrix}.$$

- For $L \gg 1$, Φ_L has strong expansion and contraction of norm $\sim L$ close to the x-direction, except near the (non-invariant) narrow strips near $x = \pm\pi/2$, where it is close to a rotation.
- It is still open whether Φ_L has positive Lyapunov exponent for any specific L.

Application: Perturbation of standard map

Blumenthal-Xue-Young considered a random perturbation of the standard map, namely for $\varepsilon > 0$, $\Phi_{L,\omega} := \Phi_L \circ S_\omega$ so that

$$D\Phi_{L,\omega} = \begin{pmatrix} L\cos(x+\omega)+2 & -1 \\ 1 & 0 \end{pmatrix},$$

where $S_\omega(x,y) = (x + \omega (\mathrm{mod} 1), y), \quad \omega \sim \mathrm{Unif}[-\varepsilon, \varepsilon]$.

Theorem (Blumenthal-Xue-Young, 2017)

For $\beta \in (0,1)$ and L large enough, if $\varepsilon \gtrsim L^{-L^{1-\beta}}$, then the top Lyapunov exponent λ_1^ε of the random dynamical system $\Phi_{L,\omega}$ with $\omega \sim \mathrm{Unif}[-\varepsilon,\varepsilon]$ satisfies

$$\lambda_1^\varepsilon \gtrsim \log L.$$

Application: Perturbation of standard map

What if we sample the maps with a different distribution? For instance can we replace $\mathrm{Unif}[-\varepsilon, \varepsilon]$ by discrete uniform measure $\mathrm{DiscUnif}_{r,\varepsilon}$ supported on $\{0, \pm\frac{1}{r}\varepsilon, \pm\frac{2}{r}\varepsilon, \dots, \pm\frac{r-1}{r}\varepsilon\}$ for some positive integer r?

Theorem (C.)

For $\delta \in (0,1)$, there exists an explicit integer $r_0 = r_0(\delta)$ such that for L large enough, if $\varepsilon \geq L^{-1+\delta}$ and $r \geq r_0(\delta)$, $\mathrm{DiscUnif}_{r,\varepsilon}$ is uniformly expanding with expansion $C \gtrsim \log L$.

Application: Perturbation of standard map

Corollary (C.)

For $\delta \in (0,1)$, there exists an explicit integer $r_0 = r_0(\delta)$ such that for L large enough, if $\varepsilon \geq L^{-1+\delta}$ and $r \geq r_0(\delta)$, then the Lyapunov exponent $\lambda_1^{\varepsilon,\mathrm{disc}}$ of the random dynamical system $\Phi_{L,\omega}$ with $\omega \sim \mathrm{DiscUnif}_{r,\varepsilon}$ satisfies

$$\lambda_1^{\varepsilon,\mathrm{disc}} \gtrsim \log L.$$

Remark: Blumenthal-Xue-Young used crucially the fact that for continuous perturbation, Lebesgue is the only stationary measure. This is not always true for discrete perturbation. In fact a consequence of our main theorem is that the only non-atomic stationary measure is Lebesgue.

General criterion for uniform expansion

Verify uniform expansion

Recall that μ is uniformly expanding if there exists $C > 0$ and $N \in \mathbb{N}$ such that for all $x \in M$ and $v \in T_x M$,

$$\int_{\mathrm{Diff}^2(M)} \log \frac{\|D_x f(v)\|}{\|v\|} d\mu^{(N)}(f) > C > 0.$$

Obstructions to uniform expansion:

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Obstructions to uniform expansion:

1. Clustering of contracting directions:
   If the contracting directions $\theta_{D_x f} \in T_x^1 M$ of a few maps $D_x f$ are "close together" on the circle $T_x^1 M$, they may get contracted "on average".

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1. Clustering of contracting directions:
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2. Rotation regions:
   On regions where the maps are close to a rotation, vectors that are expanded may get rotated to contracting directions after a few iterations.

A general criterion for uniform expansion

Theorem (C.)

For all $\lambda_{\max} > 0$, $\lambda_{\text{crit}} > 0$, and $\varepsilon > 0$ with $\varepsilon^{-3/2} \lesssim \lambda_{\text{crit}} \leq \lambda_{\max}$, there exists explicit $\eta = \eta(\lambda_{\text{crit}}, \lambda_{\max}, \varepsilon) \in (0,1)$ such that if for all $x \in M, \theta \in T^1_x M$,

$$\mu(\{f : d(\theta_{D_x f}, \theta) > \varepsilon \text{ and } \lambda_{D_x f} > \lambda_{\text{crit}}\}) > \eta,$$

and $\lambda_{D_x f} \leq \lambda_{\max}$ μ-a.s., then μ is uniformly expanding.

Intuitively, if at every point $(x, \theta) \in T^1 M$,

1. the norms of most maps are close (govern by $\lambda_{\max}, \lambda_{\text{crit}}$)
2. most maps are bounded away from a rotation (by λ_{crit}),
3. the contracting directions are "evenly" distributed (by ε),

then μ is uniformly expanding.