HYPERBOLIC GEOMETRY

1. The hyperbolic plane

The manifold.

- $\mathbb{H} = \{x + iy \mid y > 0\}$, $ds^2 = \frac{dx^2+dy^2}{y^2}$, $dA(z) = \frac{dxdy}{y^2}$.
- $\mathbb{D} = \{(r, \theta) \mid r < 1\}$, $ds^2 = 4\frac{dr^2+r^2d\theta^2}{(1-r^2)^2}$.
- $z \mapsto \frac{z-i}{z+i}$; $w \mapsto -i\frac{w+1}{w-1}$

Isometries (upper halfplane model).

- Obvious isometries
  - $z \mapsto z + x$; $N = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}$
  - $z \mapsto az$; $A = \begin{pmatrix} \sqrt{a} & \\ & 1/\sqrt{a} \end{pmatrix}$.
- Together act simply transitively. Thus $\mathrm{Isom}(\mathbb{H}) = NA \times K$ where $K = \mathrm{Stab}(i)$.
- By the disc model, $K = O(2)$. Thus $\mathrm{Isom}(\mathbb{H}) = \mathrm{PGL}_2(\mathbb{R})$ (elements of negative determinant act via $z \mapsto \frac{a\bar{z}+b}{c\bar{z}+d}$) and K acts transitively on each sphere.
- Set $G = \mathrm{PGL}_2^+(\mathbb{R}) = \mathrm{PSL}_2(\mathbb{R})$ (elements of positive determinant).

Geodesics & the boundary.

- Clearly there is a unique shortest curve connecting i, iy: the vertical line. It has length $|\log y|$. Thus the space is uniquely geodesic.
- Well-known: G maps circles and lines to circles and lines; preserves angles and boundary. Thus geodesics are lines or circles and meet boundary at right angles. This means vertical lines and semicircular arcs with endpoints on $\mathbb{R}$.
- Every geodesic segment can be infinitely extended in either direction in a unique fashion. For every distinct $x, y \in \bar{\mathbb{H}}$ there is a unique geodesic connecting them.
- Fix a point $i\infty \in \partial\mathbb{H}$. Then for every $z, z' \in \mathbb{H}$ the associated geodesic rays $\gamma(t), \gamma'(t)$ have $\lim_{t\to\infty} d(\gamma(t), \gamma'(t))$ exists. Define an equivalence relation by having the limit equal zero. A *horosphere* is an equivalence relation, clearly the set $\{z' \mid \Im(z) = \Im(z)\}$. This is an N-orbit. For the boundary point $g \cdot (i\infty)$ this is an orbit of gNg^{-1}. The horosphere is the limit of spheres of radius t around $\gamma(t)$. The region bounded by a horosphere is called a *horoball*.

Classification of isometries.

- Stablizers:
  - $\mathrm{Stab}_G(i) = K$
  - $\mathrm{Stab}_G(i\infty) = AN$
  - $\mathrm{Stab}_G(0) \cap \mathrm{Stab}_G(i\infty) = A$.
- Call $\gamma \in \mathrm{SL}_2(\mathbb{R})$

– *elliptic* if $|\mathrm{tr}(\gamma)| < 2$, equivalently if γ fixes a point in $\mathbb{H}$, or is conjugate to an element of K.
– *hyperbolic* if $|\mathrm{tr}(\gamma)| > 2$, equivalently if γ fixes a two points in $\partial\mathbb{H}$, or is conjugate to an element of A.
– *parabolic* if $|\mathrm{tr}(\gamma)| = 2$, equivalently if γ fixes a unique point in $\partial\mathbb{H}$, or is conjugate to an element of N.

2. Discrete Subgroups, fundamental domains & Cusps

Let $\Gamma < G$ be discrete, also known as a *Fuchsian group.*

Lemma 2.1. *Γ acts properly discontinuously on $\mathbb{H}$: for every compact set C, $\{\gamma \in \Gamma : \gamma C \cap C \neq \emptyset\}$ is finite.*

Definition 2.2. Let $z_0 \in \mathbb{H}$. The *Voronoi cell* of z is the set $\mathcal{F} = \{z \in \mathbb{H} \mid \forall\gamma \in \Gamma : d(z, z_0) \leq d(z, \gamma z_0)\}$.

Lemma 2.3. *$\mathcal{F}$ is convex. It is a fundamental domain* for the action of Γ. Its boundary is a countable union of geodesic segments.

Definition 2.4. Say Γ is *of the first kind* if $\int_{\mathcal{F}} dA(z) < \infty$.

Shape of $\mathcal{F}$: cusps.

Lemma 2.5. *Assume* $T = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \in \Gamma$*. Then every* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ *has either $c = 0$ or $|c| \geq 1$.*

Proof. Set $A_0 = \gamma$, $A_{n+1} = A_n T A_n^{-1}$. Then $A_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} a & a+b \\ c & c+d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} 1-ac & a^2 \\ -c^2 & 1+ac \end{pmatrix}$, and by induction can write

$$A_n = \begin{pmatrix} 1-a_n c_n & a_n^2 \\ -c_n^2 & 1+a_n c_n \end{pmatrix}$$

with $|c_n| = c^{2^n}$. Assume now that $0 < |c| < 1$. Then also $|a_n| \leq n + |a_0|$. Since $c_n \to 0$ and $a_n c_n \to 0$ it follows that $a_{n+1} = 1 - a_n c_n \to 1$ and hence that $A_n \to A$ but they are distinct. This contradicts the discreteness of Γ. □

Definition 2.6. Say $\xi \in \partial\mathbb{H}$ is a *cusp* of Γ if it is fixed by a parabolic element of Γ.

Lemma 2.7. *If ξ is a cusp then $\Gamma_\xi = \mathrm{Stab}_\Gamma(\xi)$ consists of parabolic elements.*

Proof. May assume $\xi = i\infty$ and that $T \in \Gamma$. If $A = \begin{pmatrix} a & b \\ & d \end{pmatrix} \in \Gamma$ with $|a| < 1$ then $A^n T A^{-n} = \begin{pmatrix} 1 & a^{2n} \\ & 1 \end{pmatrix} \to I_2$, a contradiction. □

Lemma 2.8. *Given a cusp ξ of Γ there exists a horoball D such that $\gamma D \cap D \neq \emptyset$ for $\gamma \in \Gamma$ implies $\gamma \in \Gamma_\xi$.*

Proof. Again assume $\xi = i\infty$ and $T \in \Gamma$. For $\gamma \notin \Gamma_\xi$ we have $|c| \geq 1$ and $\Im(\gamma z) = \frac{\Im(z)}{|cz+d|^2}$. Hence

$$\Im(z)\Im(\gamma z) = \frac{y^2}{(cx+d)^2 + c^2 y^2} \leq \frac{1}{c^2} \leq 1\,.$$

It follows that the horoball $\{y > 1\}$ has this property. □

Corollary 2.9. *All $\Gamma \backslash \Gamma_\xi$-translates of D_r $\{y > r\}$ lie in $\left\{y < r^{-1}\right\}$. In particular, for r large enough they avoid any fixed compact set.*

If ξ, ξ' are inequvivalent cusps then can also choose the horoballs to be disjoint in the quotient. It follows that the quotient has the form of a compact set together with a union of quotients of horoballs.

Corollary 2.10. *If $\Gamma\backslash\mathbb{H}$ is compact then it has no cusps.*

Note that the map $\Gamma_\xi\backslash D \to \Gamma\backslash\mathbb{H}$ is an embedding, and that the area of $\Gamma_\xi\backslash D_r$ is $\int_{-\frac{1}{2}}^{\frac{1}{2}} dx \int_1^\infty \frac{dy}{y^2} = 1$. This horobally is in fact disjoint to

Lemma 2.11. *Can bound below width of cusps; hence Γ has finitely many equivalence classes of cusps.*