PROBLEMS IN ALGEBRAIC NUMBER THEORY

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PROBLEM SET 1

Dedekind domains

1. Show that a Dedekind domain with finitely many maximal ideals is a PID.

Quadratic extensions

For a field F and $a \in F$ we define $F(\sqrt{a})$ to be the *algebra* $F[x]/\left(x^2-a\right)$. Note that this need not be a field.

2. Let d be a square-free integer. Find $\mathbb{Q}_p(\sqrt{d})$ for all p. Which cases are ramified? split? Also, find $\mathbb{R}(\sqrt{d})$.

3. Let $K = \mathbb{Q}(\sqrt{d})$. Find the ring of integers $\mathscr{O}_K$.

4. Find which rational primes p ramify, split, or are inert in the extension $K/\mathbb{Q}$.

Norms in extensions

5. Let $\mathscr{O}_K$ be a domain integrally closed in its field of fractions K, L/K a finite Galois extension of fields, $\mathscr{O}_L \subset L$ the integral closure of $\mathscr{O}_K$. For an ideal $\mathfrak{a}$ of $\mathscr{O}_L$ set $N_K^L(\mathfrak{a}) = \left(\prod_{\sigma\in\mathrm{Gal}(L/K)} \sigma(\mathfrak{a})\right) \cap K$.
   (a) Show that $N_K^L(\mathfrak{a})$ is an ideal of $\mathscr{O}_K$ and that $N_K^L(\mathfrak{a})\cdot\mathscr{O}_L = \prod_{\sigma\in\mathrm{Gal}(L/K)}\sigma(\mathfrak{a})$.
   (b) Extend this definition to separable extensions in general.
   (c) Show that $[\mathscr{O}_L;\mathfrak{a}] = \left[\mathscr{O}_K : N_K^L(\mathfrak{a})\right]$.

Local fields

Let F be a non-archimedean local field with maximal compact subring $\mathscr{O}_F$, maximal ideal $\mathfrak{p} \lhd \mathscr{O}_F$ and uniformizing element $\varpi \in \mathfrak{p}\setminus\mathfrak{p}^2$. Let $k = \mathscr{O}_F/\mathfrak{p}$ be the residue field, $q = \#k$ its order and p its characteristic. Normalize the absolute value on F so that $|\varpi| = q^{-1}$.

6. Find the domain of convergence of the series $\exp(x) \overset{\text{def}}{=} \sum_{k=0}^{\infty}\frac{x^k}{k!}$, $-\log(1-x) \overset{\text{def}}{=} \sum_{k=1}^{\infty}\frac{x^k}{k}$. Show that $\log\exp(x) = x$ and $\exp\log(1-x) = 1-x$ in appropriate discs.

7. Let V be a finite-dimensional topological vector space over F. Show that V is locally compact. Conclude that the topology on V is unique.

8. Let $\bar{F}$ be an algebraic closure of F. Show that there is a unique extension of the absolute value to $\bar{F}$. Show that $\bar{F}$ is not complete with respect to this absolute value. Show that the metric completion of $\bar{F}$ is algebraically closed. The completion of the algebraic closure of $\mathbb{Q}_p$ is denoted $\mathbb{C}_p$.

Let $\bar{\mathfrak{p}} \lhd \bar{\mathscr{O}} \subset \bar{F}$ be defined using that absolute value, and let $\bar{k} = \bar{\mathscr{O}}/\bar{\mathfrak{p}}$.

9. Show that $\bar{k}$ is an algebraic closure of k.