Math 422/501: Problem set 2 (due 23/9/09)

Direct and semidirect products

1. Let G be a group, and let A,B be subgroups of G so that B is normal and $A \cap B = \{e\}$.
   (a) Show that $A \ltimes B = \{a \cdot b \mid a \in A,\, b \in B\}$ is a subgroup of G; and that every element of it can be uniquely written as a product $a \cdot b$. We call this subgroup the *internal semidirect product* of A,B.
   (b) Assuming that A is normal as well show that $ab = ba$ for all $a \in A$, $b \in B$. In that case we say that the subgroup AB is the *internal direct product* of A,B.

2. Let G,H be groups. Let $G \times H = \{(g,h) \mid g \in G,\, h \in H\}$ and give it the group structure $(g,h) \cdot (g',h') = (gg',hh')$. Show that this makes $G \times H$ into a group (called the *direct product* of G,H) and find normal subgroups $\bar{G},\bar{H} < G \times H$ isomorphic to G,H respectively so that $G \times H$ is the internal direct product of $\bar{G}$ and $\bar{H}$.

3. Let G,H be groups and let G act on H by automorphisms (in other words, for each $g \in G$ you are given a group isomorphism $\alpha_g \colon H \to H$ such that $\alpha_{gh} = \alpha_g \circ \alpha_h$). Give the set $G \times H$ the group structure $(g',h') \cdot (g,h) = \left(g'g, \alpha_{g^{-1}}(h')h\right)$. Show that this gives a group structure called the *semidirect product* $G \ltimes H$. Show that the semidirect product contains subgroups $\bar{G},\bar{H}$ with $\bar{H}$ normal such that $G \ltimes H$ is the internal semidirect product of G, H.

p-Groups

4. Let G be a non-abelian group of order p^3, p a prime. Show that $Z(G)$ has order p and that $G/Z(G) \simeq C_p \times C_p$.

Cyclic group actions and cycle decompositions

5. Let G be a group acting on a set X, and let $g \in G$. Show that a subset $Y \subset X$ is invariant under the action of the subgroup $\langle g \rangle$ of G iff $gY = Y$. When Y is finite show that assuming $gY \subset Y$ is enough.

6. For $\alpha \in S_n$ write $\operatorname{supp}(\alpha)$ for the set $\{i \in [n] \mid \alpha(i) \neq i\}$.
   (a) Show that $\operatorname{supp}(\alpha)$ is invariant under the action of $\langle \alpha \rangle$.
   (b) Show that if $\operatorname{supp}(\alpha) \cap \operatorname{supp}(\beta) = \emptyset$ then $\alpha\beta = \beta\alpha$.

7. (Cycle decomposition) Call $\sigma \in S_n$ a *cycle* if its support is a single orbit of $\langle\sigma\rangle$, in which case we call the size of the support the *length* of the cycle.
   (a) Let $\alpha \in S_n$, and let $O \subset [n]$ be an orbit of $\langle\alpha\rangle$ of length at least 2. Show that there exists a unique cycle $\beta \in S_n$ supported on O so that $\alpha \restriction_O = \beta \restriction_O$ (that is, the restrictions of the functions α, β to the set O are equal).
   (b) Let $\alpha \in S_n$ and let $\{\beta_O \mid O \text{ an orbit of } \langle\alpha\rangle\}$ be the set of cycles obtained in part (a). Show that they all commute and that their product is α.
   (c) Show that every element of S_n can be written uniquely as a product of cycles of disjoint support.
   (d) Consider the action of $[4]_{35} = 4 + 35\mathbb{Z} \in \mathbb{Z}/35\mathbb{Z}$ by multiplication on $\mathbb{Z}/35\mathbb{Z}$. Decompose this permutation into a product of cycles.

8. (The conjugacy classes of S_n)
   (a) Let $\alpha, \beta \in S_n$ with α a cycle. Show that $\beta\alpha\beta^{-1}$ is a cycle as well.
   (b) Show that $\alpha, \beta \in S_n$ are conjugate iff for each $2 \le l \le n$ the number of cycles of length l in their cycle decomposition is the same.
   Hint: Constructs a bijection from $[n]$ to $[n]$ that converts one partition into orbits into the other.

Affine algebra

DEFINITION 66. Let F be a field, V/F a vector space. An *affine combination* is a formal sum $\sum_{i=1}^n t_i \underline{v}_i$ where $t_i \in F$, $\underline{v}_i \in V$ and $\sum_{i=1}^n t_i = 1$. If V, W are vector spaces then a map $f\colon V \to W$ is called an *affine map* if for every affine combination in V we have

$$f\left(\sum_{i=1}^n t_i \underline{v}_i\right) = \sum_{i=1}^n t_i f(\underline{v}_i) .$$

9. (The affine group) Let U, V, W be vector spaces over F, $f\colon U \to V$, $g\colon V \to W$ affine maps.
   (a) Show that $g \circ f\colon U \to W$ is affine.
   (b) Assume that f is bijective. Show that its set-theoretic inverse $f^{-1}\colon V \to U$ is an affine map as well.
   (c) Let $\mathrm{Aff}(V)$ denote the set of invertible affine maps from V to V. Show that $\mathrm{Aff}(V)$ is a group, and that it has a natural action on V.
   (d) Assume that $f(\underline{0}_U) = \underline{0}_V$. Show that f is a linear map.
10. (Elements of the affine group)
   (a) Given $\underline{a} \in V$ show that $T_{\underline{a}}\underline{x} = \underline{x} + \underline{a}$ ("translation by $\underline{a}$") is an affine map.
   (b) Show that the map $\underline{a} \mapsto T_{\underline{a}}$ is a group homomorphism from the additive group of V to $\mathrm{Aff}(V)$. Write $\mathbb{T}(V)$ for the image.
   (c) Show that $\mathbb{T}(V)$ acts transitively on V. Show that the action is *simple*: for any $\underline{x} \in V$, $\mathrm{Stab}_{\mathbb{T}(V)}(\underline{x}) = \{T_{\underline{0}}\}$.
   (d) Fixing a basepoint $\underline{0} \in V$, show that every $A \in \mathrm{Aff}(V)$ can be uniquely written in the form $A = T_{\underline{a}}B$ where $\underline{a} \in V$ and $B \in \mathrm{GL}(V)$. Conclude that $\mathrm{Aff}(V) = \mathbb{T}(V) \cdot \mathrm{GL}(V)$ setwise.
   (e) Show that $\mathbb{T}(V) \cap \mathrm{GL}(V) = \{1\}$ and that $\mathbb{T}(V)$ is a normal subgroup of $\mathrm{Aff}(V)$. Show that $\mathrm{Aff}(V)$ is isomorphic to the semidirect product $\mathrm{GL}(V) \ltimes (V, +)$.

Additional (not for credit)

A. Let F be a finite field with q elements, V/F a vector space of dimension n. Find a formula for the *Gaussian binomial coefficient* $\binom{n}{k}_q$, the number of k-dimensional subspaces of V of dimension k. Show that this is a polynomial in q and that its limit as $q \to 1$ is the usual binomial coefficient $\binom{n}{k}$.

10. Let F be a field, V a finite-dimensional F-vector space. A *flag* in V is a nested sequence $\{0\} = W_0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_k \subsetneq W_{k+1} = V$ of subspaces of V.
    (a) Show that $G = \mathrm{GL}(V)$ acts on the space for flags.
    (b) Find the orbits of the action and show that no two are isomorphic as G-sets. Orbit stabilizers are called *parabolic subgroups*.
    (c) Let F be finite (say with q elements). Find the size of each orbit.
    Hint: The set of subspaces of V containing W is in bijection with the set of subspaces of the quotient vector space V/W.
    (d) Let $B < G$ be the stabilizer of a maximal flag ("Borel subgroup"). Find the order of B, hence the order of G.