Math 422/501: Problem set 4 (due 7/10/09)

Solvable groups

1. Show the following are equivalent:
   (a) Every finite group of odd order is solvable.
   (b) Every non-abelian finite simple group is of even order.
   Aside: That (a) holds is a famous Theorem of Feit and Thompson (1963).

2. Let F be a field. Let $G = \mathrm{GL}_n(F)$, let $B < G$ be the subgroup of upper-triangular matrices, $N < B$ the subgroup of matrices with 1s on the diagonal. Next, for $0 \le j \le n-1$ write N_j for the matrices with $1s$ on the main diagonal and 0s on the next j diagonals. When $n = 4$ we have:
$$N = N_0 = \left\{ \begin{pmatrix} 1 & * & * & * \\ & 1 & * & * \\ & & 1 & * \\ & & & 1 \end{pmatrix} \right\}, N_1 = \left\{ \begin{pmatrix} 1 & 0 & * & * \\ & 1 & 0 & * \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} \right\}, N_2 = \left\{ \begin{pmatrix} 1 & 0 & 0 & * \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} \right\} \text{ etc}$$
.
   (a) Show that $N \lhd B$ and that $B/N \simeq (F^\times)^n$ (direct product of n copies).
   (b) For each $0 \le j < n-1$, $N_{j+1} \lhd N_j$ and $N_j/N_{j+1} \simeq F^{n-j-1}$ (direct products of copies of the additive group of F).
   (c) Conclude that B is solvable.

DEFINITION. Let G be a group. The *derived series* of G is the sequence of subgroups defined by $G^{(0)} = G$ and $G^{(i+1)} = \left(G^{(i)}\right)'$ (commutator subgroups).

3. Let G be a group and assume $G^{(k)} = \{e\}$. Show that G is solvable.

4. Let G be a solvable group, say with normal series $G = G_0 \rhd G_1 \rhd G_2 \rhd \cdots \rhd G_k = \{e\}$. Show that $G^{(i)} < G_i$ for all i. Conclude that G is solvable iff the derived series terminates.

OPTIONAL Let $S_\infty \subset S_{\mathbb{N}}$ denote the set of permutations of *finite support*.
   (a) Show that $S_\infty = \bigcup_n S_n$ with respect to the natural inclusion of $S_n = S_{[n]}$ in S_∞.
   (b) Let $A_\infty = \bigcup_n A_n$ with respect to the same inclusion. Show that A_∞ is a subgroup of S_∞ of index 2.
   (c) Show that A_∞ is simple.