Math 437/537: Problem set 6 (due 4/12/09)

Prime estimates

1. In class we found $0 < \delta < 1 < \Delta$ so that $\delta x \leq \nu(x) \leq \Delta x$ for $x \geq 2$. Complete the proof of Chebychev's Theorem by finding $0 < A < B$ so that $A\frac{x}{\log x} \leq \pi(x) \leq B\frac{x}{\log x}$ if $x \geq 2$.

2. Find $0 < C < D$ so that $C \log\log x \leq \sum_{p \leq x} \frac{1}{p} \leq D \log\log x$ for $x \geq 3$.
   Hint: Break the range of summation into dyadic intervals $\left[2^j \leq p < 2^{j+1}\right]$.

OPT (The average number of prime divisors) Let $P(x) = \frac{1}{x}\sum_{n \leq x} \omega(n)$.
   (a) Show that $P(x) = \frac{1}{x}\sum_{p \leq x} \left\lfloor \frac{x}{p} \right\rfloor$ (sum over primes).
   Hint: Write $\omega(n) = \sum_{p|n} 1$ and change the order of summation.
   (b) Show that $C \log\log x - 1 \leq P(x) \leq D \log\log x$ for $x \geq 3$.
   Hint: $y - 1 \leq \lfloor y \rfloor \leq y$.
   (c) Mertens has found E so that $\left|\sum_{p \leq x} \frac{1}{p} - \log\log x\right| \leq E$ for all x. Conclude that $|P(x) - \log\log x|$ is uniformly bounded as well.
   — This result is usually phrased: "the average number of distinct primes dividing a random integer is about $\log\log x$".

Irrationality and continued fracionts

3. Show that the following numbers are irrational:
   (a) $\frac{\log n}{\log m}$ where $n, m \geq 2$ are relatively prime integers.
   (b) $e = \sum_{n=0}^{\infty} \frac{1}{n!}$.
   Hint: Consider $\lfloor k! e \rfloor$.
   (c) $\sum_{n=0}^{\infty} \frac{1}{3^{4^n}}$.
   Hint: Multiply by a power of 3 and consider the fractional part.

OPT (Egyptian fractions) Show that $r \in \mathbb{Q} \cap (0,1)$ can be written in the form $r = \sum_{i=1}^{t} \frac{1}{q_i}$ with distinct $q_i \in \mathbb{Z}_{>0}$.

4. (Hermite) Let p be a prime such that $p \equiv 1\,(4)$. Let $0 < u < p$ with $u^2 \equiv -1\,(p)$. Write $\frac{u}{p} = \langle a_0, \ldots, a_n \rangle$ and let i be maximal such that $k_i \leq \sqrt{p}$.
   (a) Show that $\left|\frac{h_i}{k_i} - \frac{u}{p}\right| < \frac{1}{k_i\sqrt{p}}$. Conclude that $|h_i p - u k_i| < \sqrt{p}$.
   (b) Let $x = k_i$, $y = h_i p - u k_i$. Show that $0 < x^2 + y^2 < 2p$. Show that $x^2 + y^2 \equiv 0\,(p)$ and conclude that $p = x^2 + y^2$.
5. Calculate the 0th through 4th convergents to π.

6. (Convergence)
   (a) Suppose the infinite continued fractio n exapnsions of θ, η agree through a_n. Show that

   $|\theta - \eta| \leq \frac{1}{k_n^2}$.

(b) Show that $\lim_{n\to\infty}\langle a_0,a_1,\ldots,a_n,b_{n+1},b_{n+2},\ldots\rangle = \langle a_0,a_1,\ldots\rangle$.

The continued fraction expansion of e.

Set $(-1)!! = 0!! = 1$ and for $n\geq 1$,

$$n!! = \prod_{\substack{1\leq j\leq n\\ j\equiv n\,(2)}} j\,.$$

Now for $n\geq 0$ set:

$$\psi_n(x) = \sum_{k=0}^{\infty}\frac{x^{2k}}{(2k+2n-1)!!(2k)!!}\qquad ,\, w_n(n) = \frac{\psi_n(x)}{x\psi_{n+1}(x)}\,.$$

7. (Evaluation)
   (a) Show that $\psi_n(x)$ are entire functions.
   (b) Show that $\psi_0(x) = \cosh(x) = \frac{e^x+e^{-x}}{2}$ and that $\psi_1(x) = \frac{\sinh(x)}{x} = \frac{e^x-e^{-x}}{2x}$. Conclude that $w_0(x) = \tanh(x) = \frac{e^x+e^{-x}}{e^x-e^{-x}}$.
   (d) Show that $\psi_n(x) = (2n+1)\psi_{n+1}+x^2\psi_{n+2}$. Conclude that $w_n(x) = \frac{2n+1}{x}+\frac{1}{w_{n+1}(x)}$.
   (e) Using your answer to part (d) show that $\frac{e^{1/k}+e^{-1/k}}{e^{1/k}-e^{-1/k}} = \langle k,3k,5k,7k,9k,\cdots\rangle$ for all $k\geq 1$.

8. (Calculation)
   (a) Let $u = w_0(\frac{1}{2})$ and let $v = \langle v_0,v_1,v_2,v_3,\ldots\rangle$ where $v_0 = 0$, $v_1 = 5 = 2\cdot(2\cdot 1+1)-1$ and $v_n = 2(2n+1)$ for $n\geq 2$. Show that $u = 2+\frac{1}{1+\frac{1}{v}}$
   (b) Show that $e = \frac{u+1}{u-1} = \langle 2,1+2v\rangle$.
   (c) Let ξ be a real number, $b\geq 2$ an integer, and let $\alpha = \langle 0,2b–1,\xi\rangle$. Show that $2\alpha = \left\langle 0;b-1,1,1+\frac{2}{\xi-1}\right\rangle$.
   (d) Let $\{b_n\}_{n=1}^{\infty}\subset\mathbb{Z}_{\geq 2}$ and let $\alpha = \langle 0,2b_1-1,2b_2,2b_3,\ldots\rangle$. Show that
$$2\alpha = \langle 0,b_1-1,1,1,b_2-1,1,1,b_3-1,1,1,\ldots\rangle\,.$$

9. (Punchline) Show that
$$e = \langle 2,1,2,1,1,4,1,1,6,1,1,8,1,1,10,\cdots\rangle = \langle 2,1,e_2,e_3,e_4,\cdots\rangle$$
where $e_n = \begin{cases}2k & n=3k-1\\ 1 & n\equiv 0,1\,(3)\end{cases}$.