Math 613: Problem set 2 (due 22/9/09)

Let V be an n-dimensional inner product space, and fix a lattice $\Lambda < \mathbb{R}^n$ with basis $\left\{v_j\right\}_{j=1}^n$. Write $\mathbb{T} = V/\Lambda$ for the quotient torus, a compact space.

Integration on $\mathbb{T}$

DEFINITION. A *fundamental domain* for Λ is a closed subset $\mathcal{F} \subset V$ such that:

(1) $\cup_{v\in\Lambda}(v+\mathcal{F}) = V$, that is $\mathcal{F}$ intersects every orbit and surjects on $\mathbb{T}$.
(2) There is an open set $\mathcal{F}^\circ$ which injects into $\mathbb{T}$ and such that $\mathcal{F} = \overline{\mathcal{F}^\circ}$.
(3) The difference set $\mathcal{F} \setminus \mathcal{F}^\circ$ has measure zero.

1. Show that $\mathcal{F}_{\frac{1}{2}} = \left\{\sum_{j=1}^n a_j v_j \mid \left|a_j\right| \le \frac{1}{2}\right\}$ and $\mathcal{F}_1 = \left\{\sum_{j=1}^n a_j v_j \mid 0 \le a_j \le 1\right\}$ are fundamental domains.

*2. (The Dirichlet domain) Fix $x_0 \in V$ and set

$$\mathcal{F}_{\mathrm{D}} = \left\{x \in V \mid \forall v \in \Lambda : \|x - x_0\| \le \|x - (x_0 + v)\|\right\}.$$

(a) Show that $\mathcal{F}_{\mathrm{D}}$ is closed and surjects on $\mathbb{T}$.
Hint: Write it as an intersection of closed half-spaces.
(b) Show that $\mathcal{F}_{\mathrm{D}}$ is bounded.
Hint: Show that $\mathcal{F}_{\mathrm{D}} \subset B(x_0, 2\,\mathrm{diam}(\mathcal{F}_1))$.
(c) Show that $\mathcal{F}_{\mathrm{D}}$ is the intersection of finitely many closed half-spaces.
(d) Let $\mathcal{F}_{\mathrm{D}}^\circ$ be the intersection of the interiors of these half-spaces and show that $\mathcal{F}_{\mathrm{D}}$ is a fundamental domain.

3. (Lattice averaging) A function $f \in C(V)$ is said to be *of rapid decay* if for all $N \ge 1$ the function $(1 + \|x\|)^N f(x)$ is bounded. $f \in C^\infty(V)$ is said to be of *Schwartz class* if it and all its derivatives are of rapid decay (the set of such functions is denoted $\mathcal{S}(V)$).

(a) Let f be of rapid decay. Show that for all $x \in V$, $(\Pi_\Lambda f)(x) \stackrel{\text{def}}{=} \sum_{v\in\Lambda} f(x+v)$ converges and defines a continuous function on $\mathbb{T}$.
OPT Let $f \in \mathcal{S}(V)$. Show that $\Pi_\Lambda f$ is smooth.
(b) (Smooth fundamental domain) Let $\chi_0 \in C_c^\infty(V)$ be non-negative and satisfy $\chi_0 \restriction_{\mathcal{F}} \equiv 1$ for some compact fundamental domain $\mathcal{F}$. Show that $\chi(x) = \frac{\chi_0(x)}{(\Pi_\Lambda \chi_0)(x)} \in C_c^\infty(V)$ and that we have $\Pi_\Lambda \chi \equiv 1$.

*4. (Integration on $\mathbb{T}$) dx will denote the Lebesgue measure on V. For $f \in C(\mathbb{T})$ define $\int_{\mathbb{T}} f(\bar{x})d\bar{x} = \int_V f(x)\chi(x)dx$.

(a) Show that the integral on the RHS defines a linear map $C(\mathbb{T}) \to \mathbb{C}$ mapping non-negative functions to non-negative reals.
(b) Show that for any fundamental domain $\mathcal{F}'$ for Λ we have

$$\int_{\mathcal{F}'} f(x)dx = \int_V f\chi.$$

(c) Conclude that the measure $d\bar{x}$ on $\mathbb{T}$ is translation-invariant.

(d) Show that the volume of $\mathbb{T}$ is the absolute value of the determinant of the matrix A such that a_{ij} is the ith co-ordinate of v_j in an orthonormal basis of V. Conclude that $(\mathrm{vol}(\mathbb{T}))^2$ is the determinant of the *Gram matrix*, whose ijth entry is $\langle v_i, v_j \rangle$.
Hint: Use one of the fundamental domains of problem 1.

5. Let V be an n-dimensional real vector space, V^* the dual space. Let $\Lambda < V$ be a lattice, and set $\Lambda^* = \{v^* \in V^* \mid v^*(\Lambda) \subset \mathbb{Z}\}$.
(a) Show that Λ^* is a lattice in V^*.
Hint: Use the dual basis.
(b) Show that the standard isomorphism $V \simeq V^{**}$ identifies Λ with Λ^{**}.
(*c) If V is an inner product space show that $\mathrm{vol}(V/\Lambda)\,\mathrm{vol}(V^*/\Lambda^*) = 1$.

The Poisson Summation Formula

We will use the standard notation $e(z) = \exp(2\pi i z)$. For a short note on Fourier series and the Poisson Summation Formula see the course website.

6. Fix $k \geq 2$.
(a) Show that $\tau \mapsto \sum_{d\in\mathbb{Z}} \frac{1}{(\tau+d)^k}$ is holomorphic in $\mathbb{H} = \{x+iy \mid y > 0\}$.
(b) Show that
$$\int_{\mathbb{R}} \frac{e(-rx)}{(x+\tau)^k}dx = \begin{cases} \frac{(-2\pi i)^k}{(k-1)!} r^{k-1} e(r\tau) & r \geq 0 \\ 0 & r \leq 0 \end{cases}$$
(c) Show that $\sum_{d\in\mathbb{Z}} \frac{1}{(\tau+d)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{m=1}^{\infty} m^{k-1} e(m\tau)$.
(d) Show that there exists C such that
$$\frac{1}{\tau} + \sum_{d\geq 1} \left(\frac{1}{\tau+d} + \frac{1}{\tau-d} \right) = C + (-2\pi i) \sum_{m=0}^{\infty} e(m\tau) = C - \frac{2\pi i}{1-e(\tau)}.$$
Hint: After showing that both sides are differentiable, take their derivatives.
(e) Multiply by τ and use the Taylor expansion of both sides to show that $C = \pi i$ and that
$$1 - 2\sum_{k=1}^{\infty} \zeta(2k)\tau^{2k} = \pi i \tau \frac{e(\tau)+1}{e(\tau)-1}.$$
Hint: $\sum_{d\geq 1} \frac{1}{d^2} = \frac{\pi^2}{6}$.
(f) Show that for $k \geq 1$ even, $\zeta(k) = -\frac{1}{2}\frac{(2\pi i)^k}{k!} B_k$ where B_k are rational numbers.
Hint: let $\frac{t}{2}\frac{e^t+1}{e^t-1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} t^k$.

7. let $\varphi(x) = \exp\left\{-\pi\alpha \|x\|^2\right\}$ on $\mathbb{R}^n$ where $\Re(\alpha) > 0$.
(a) Show that $\hat{\varphi}(k) = \alpha^{-n/2} \exp\left\{-\frac{\pi}{\alpha}\|k\|^2\right\}$ (take the branch of the square root defined on $\Re(\alpha) > 0$ such that $\sqrt{1} = 1$).
(b) Conclude that $\theta(\tau) = \sum_{n\in\mathbb{Z}} e^{2\pi i n^2 \tau}$ satisfies $\theta(-\frac{1}{4\tau}) = \sqrt{-2i\tau}\,\theta(\tau)$

Continuing the Epstein Zetafunction

8. For $\varphi \in C^\infty(V)$ of Schwartz class set $\varphi(\Lambda) = \sum'_{v\in\Lambda} \varphi(v)$ and $Z(\Lambda;\varphi;s) = \int_0^\infty \varphi(r\Lambda) r^{ns} \frac{dr}{r}$.
   (a) Show that the sum converges absolutely.
   (b) Show that as $r \to \infty$, $|\varphi|(r\Lambda)$ decays faster than any polynomial and that as $r \to 0$, $|\varphi|(r\Lambda) = O(r^{-n})$. Conclude that $Z(\Lambda;\varphi;s)$ converges absolutely in $\Re(s) > 1$ and defines a holomorphic function there.
   (c) Applying the Poisson summation formula, show that for $\Re(s) > 1$,

$$Z(\Lambda;\varphi;s) = \int_1^\infty \varphi(r\Lambda) r^{ns} \frac{dr}{r} - \varphi(0)\frac{1}{ns} + \frac{1}{\mathrm{vol}(\Lambda)} \int_1^\infty \hat{\varphi}(r\Lambda^*) r^{n(1-s)} \frac{dr}{r} - \frac{\hat{\varphi}(0)}{\mathrm{vol}(\Lambda)} \frac{1}{n(1-s)}.$$

   (d) Since $\varphi \in \mathcal{S}(V)$ we also have $\hat{\varphi} \in \mathcal{S}(V)$ and $\hat{\hat{\varphi}}(x) = \varphi(-x)$. Conclude that $Z(\Lambda;\varphi;s)$ extends to a meromorphic function of s with poles at $s = 0, 1$ which satisfies the *functional equation*

$$\sqrt{\mathrm{vol}(\Lambda)} Z(\Lambda;\varphi;s) = \sqrt{\mathrm{vol}(\Lambda^*)} Z(\Lambda^*;\hat{\varphi};1-s).$$

   (e) Assume that φ is spherical, and show that for $\Re(s) > 1$ we have

$$Z(\Lambda;\varphi;s) = \left(\int_0^\infty \varphi(r) r^{ns} \frac{dr}{r}\right) E(\Lambda;s).$$

   (f) For $\varphi \in C_c^\infty(\mathbb{R}_{>0}^\times)$, show that $\left(\int_0^\infty \varphi(r) r^{ns} \frac{dr}{r}\right)$ extends to an entire function; conclude that $E(\Lambda;s)$ extends to a meromorphic function of s.
   (g) For $\varphi(x) = \exp\left\{-\pi \|x\|^2\right\}$ show that $\varphi(x) = \hat{\varphi}(x)$ and that

$$\int_0^\infty \varphi(r) r^{ns} \frac{dr}{r} = 2\pi^{-ns/2} \Gamma(\frac{ns}{2}).$$

   Conclude that $E(\Lambda;s)$ has no poles other than $s = 0, 1$ and satisfies the functional equation

$$\sqrt{\mathrm{vol}(\Lambda)} \pi^{-\frac{n}{2}\left(\frac{1}{2}+s\right)} \Gamma(\frac{ns}{2}) E(\Lambda;s) = \sqrt{\mathrm{vol}(\Lambda^*)} \pi^{-\frac{n}{2}\left(\frac{1}{2}-s\right)} \Gamma(\frac{n(1-s)}{2}) E(\Lambda^*;1-s).$$

   (h) Show from (c) that $\mathrm{Res}_{s=1} E(\Lambda;s) = \frac{\pi^{n/2}}{2\,\mathrm{vol}(\Lambda)\Gamma\left(\frac{n}{2}\right)}$.

REMARK 50. We often write $\mathrm{vol}(\Lambda)$ for the *covolume* $\mathrm{vol}(V/\Lambda)$.