Math 613: Problem set 3 (due 4/10/09)

For a group G acting on a space X write $G\backslash X$ for the space of orbits. If X is a topological space, G a topological group and the action $G \times X \to X$ is continuous we endow $G\backslash X$ with the quotient topology.

The moduli space of elliptic curves

1. Let $T = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$, $S = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$ and let $\Gamma < \mathrm{SL}_2(\mathbb{Z})$ be the subgroup they generate, Γ_∞ the subgroup generated by $\pm T$ (note that $-I$ acts trivially on the upper half-plane). Let $\mathcal{S}$ denote the strip $\left\{|\Re(\tau)| \le \frac{1}{2}\right\}$ and let $\mathcal{F} = \left\{\tau \in \mathbb{H} \mid |\tau| \ge 1, |\Re(\tau)| \le \frac{1}{2}\right\}$.
   (a) Show that $\mathcal{S}$ is a fundamental domain for $\Gamma_\infty\backslash\mathbb{H}$, hence surjects on $\Gamma\backslash\mathbb{H}$.
   (b) Let $\tau = x + iy \in \mathcal{S}$. Show that there are only finitely many $y' \ge y$ such that there exists x' for which $\tau' = x' + iy' \in \mathrm{SL}_2(\mathbb{Z}) \cdot \tau$.
   Hint: Recall that $y(\gamma\tau) = \frac{y(\tau)}{|c\tau+d|^2}$, and consider the real and imaginary parts of $c\tau + d$ separately.
   (c) Let $f \colon \mathcal{S} \to \mathcal{S}$ be as follows: if $|\tau| \ge 1$ set $f(\tau) = \tau$. Otherwise, let $f(\tau) = T^m S\tau$ with m chosen so that $f(\tau) \in \mathcal{S}$. Show that $\Im(f(\tau)) > \Im(\tau)$.
   (d) Conclude that $\mathcal{F}$ surjects on $\Gamma\backslash\mathbb{H}$.
   (e) Let $\tau \in \mathcal{F}$ and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ be such that $\gamma\tau \in \mathcal{F}$ but $\gamma \ne \pm I$. Show that one of the following holds:
      (1) $|\Re(\tau)| = \frac{1}{2}$ and $\gamma \in \left\{\pm T, \pm T^{-1}\right\}$.
      (2) $|\tau| = 1$ and $\gamma \in \{\pm S\}$.
      (3) $\tau = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$
   (f) Show that $-I \in \Gamma$ and conclude that $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and that $\mathcal{F}$ is a fundamental domain for $Y(1) = \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$.

OPT Let $E = \mathbb{C}/\Lambda$ be an elliptic curve.
   (a) Show that up to isomorphism of elliptic curves we may assume that $1 \in \Lambda$ and that it is a non-zero element of minimal length.
   (b) Let $\tau \in \mathbb{H} \cap \Lambda$ be of minimal norm. Show that $|\tau| \ge 1$ and that $|\Re(\tau)| \le \frac{1}{2}$, that is that $\tau \in \mathcal{F}$.
   (c) Show for any $z \in \mathbb{C}$ there is $z' \in z + \Lambda_\tau$ with $|z'| < \frac{1}{2} + \frac{1}{2}|\tau| \le |\tau|$ and conclude that $\Lambda = \Lambda_\tau$, that is that $\mathcal{F}$ surjects on $Y(1)$.
   – Using 1(e) it follows again that $\mathcal{F}$ is a fundamental domain.

3. Let $dA(\tau) = \frac{dxdy}{y^2}$ denote the hyperbolic area measure on $\mathbb{H}$. Calculate $\int_{\mathcal{F}} dA(\tau)$.

The moduli space of elliptic curves with level structure

4. Let $\Lambda < \mathbb{C}$ be a lattice, $E = \mathbb{C}/\Lambda$ the associated elliptic curve. For an integer N write $E[N]$ for the *N-torsion points*, that is the points $x \in E$ such that $N \cdot x = 0$.
   (a) Show that $E[N] \simeq (\mathbb{Z}/N\mathbb{Z})^2$ as abelian groups.
   – We now study the action of $G = \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ on $E[N]$.
   (b) Show that G acts transitively on the set of points in $E[N]$ whose order is N exactly. Find the stabilizer of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (call it $K_1(N)$) and the number of such points.
   (c) Conclude that G acts transitively on the set of subgroups of $E[N]$ which is cyclic of order N. Find the stabilizer of the subgroup $\left\{ \begin{pmatrix} * \\ 0 \end{pmatrix} \right\}$ (call it $K_0(N)$) and the number of such subgroups.
   (d) Find the order of $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$. Write in the form $N^3 \prod_{p|N} f(p)$.

5. Let $Y_0(N)$ denote the set of isomorphism classes of pairs (E,C) where E is a complex elliptic curve and $C \subset E$ is a subgroup isomorphic to C_N ($(E,C) \sim (E,C')$ if there exists an isomorphism $f\colon E \to E'$ such that $f(C) = C'$).
   (a) Show that the map $\mathbb{H} \to Y_1(N)$ mapping τ to the class of the pair $\left(\mathbb{C}/\Lambda_\tau, \frac{1}{N}\mathbb{Z}/\mathbb{Z}\right)$ (i.e. the subgroup of $\mathbb{C}/\Lambda_\tau$ generated by $\frac{1}{N} + \Lambda_\tau$) is surjective.
   (b) By analyzing the isomorphism relation show that $Y_0(N) = \Gamma_0(N)\backslash\mathbb{H}$ where $\Gamma_0(N)$ is the inverse image in $\mathrm{SL}_2(\mathbb{Z})$ of $K_0(N)$.

OPT Let $Y_1(N)$ denote the set of isomorphism classes of pairs (E,P) where E is a complex elliptic curve and $P \in E[N]$ has order N exactly.
   (a) Show that the map $\mathbb{H} \to Y_1(N)$ mapping τ to the class of the pair $\left(\mathbb{C}/\Lambda_\tau, \frac{1}{N} + \Lambda_\tau\right)$ is surjective.
   (b) By analyzing the isomorphism relation show that $Y_1(N) = \Gamma_1(N)\backslash\mathbb{H}$ where $\Gamma_1(N)$ is the inverse image in $\mathrm{SL}_2(\mathbb{Z})$ of $K_0(N)$.

OPT Let $Y(N)$ denote the set of isomorphism classes of triples (E,P,Q) where E is a complex elliptic curve and $P,Q \in E[N]$ are an ordered basis for $E[N]$ as a free $\mathbb{Z}/N\mathbb{Z}$-module.
   (a) Show that the map $\mathbb{H} \to Y(N)$ mapping τ to the class of the triple $\left(\mathbb{C}/\Lambda_\tau, \frac{1}{N}\mathbb{Z} + \Lambda_\tau, \frac{\tau}{N} + \Lambda_\tau\right)$ is surjective.
   (b) By analyzing the isomorphism relation show that $Y(N) = \Gamma(N)\backslash\mathbb{H}$ where $\Gamma(N)$ is the kernel of the map $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$.

Hyperbolic Convergence Lemma

Let $\Gamma < \mathrm{SL}_2(\mathbb{R})$ be discrete and assume that $\Gamma_\infty = \Gamma \cap P$ is non-trivial (i.e. infinite), with the image in $\mathrm{PSL}_2(\mathbb{R})$ generated by $\begin{pmatrix} 1 & h \\ & 1 \end{pmatrix}$.

8. (Counting Lemma)
   (a) Show that a fundamental domain for $\Gamma_\infty \backslash \mathbb{H}$ is the strip $\left\{|\Re(z)| \leq \frac{h}{2}\right\}$.
   (b) Calculate the hyperbolic area of the half-strip $\left\{x+iy \mid |x| \leq \frac{h}{2}, y \geq \frac{1}{Y}\right\}$.
   (c) For $z \in \mathbb{H}$ show that there exists $C > 0$ (depending locally uniformly on z) such that for all $Y > 0$, $\#R_Y \leq C(1+Y)$ where
   $$R_Y = \left\{\Gamma_\infty \gamma \in \Gamma_\infty \backslash \Gamma \mid y(\gamma z) \geq \frac{1}{Y}\right\}.$$
   Hint: Let B be a hyperbolic ball around z of small enough radius so that if $\gamma \in \Gamma$ satisfies $\gamma B \cap B \neq \emptyset$ then γ belongs to the finite group Γ_z, and consider the set of images of $\Gamma \cdot B$ in the strip.

For $\Re(s) > 1$ we define the *non-holomorphic Eisenstein series* to be

$$E(z;s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} y(\gamma z)^{-s}$$

9. (Convergence Lemma)
   (a) Show that the series $E(z;\sigma)$ converges absolutely if $\sigma > 1$.
   Hint: Show that $E(\sigma;z) \leq A + \sum_{n=1}^{\infty} (\#R_{n+1} - \#R_n)\, n^{-\sigma}$ where A is easily controlled. Now use summation by parts.
   (b) Conclude that $E(z;s)$ extends to a holomorphic function of s in $\Re(s) > 1$.