Modular Forms
Lecture Notes

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These are rough notes for the fall 2010 course. Problem sets and solutions were posted on an internal website.

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Preface

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0.1. What is a modular form?

- Holomorphic function satisfying a periodicity condition.
- Generating function for an arithemetic function.
- Section of a line bundle on a 1-dimensional complex manifold
- Section of an algebraic line bundle on a 1-dimensional complex algebraic variety
- Solution to a PDE satisfying certain boundary conditions
- Irreducible representation of $\mathrm{SL}_2(\mathbb{R})$ acting on $L^2(\Gamma\backslash\mathrm{SL}_2(\mathbb{R}))$
- Irreducible representation of $\mathrm{SL}_2(\mathbb{A})$ acting on $L^2(\mathrm{SL}_2(\mathbb{Q})\backslash\mathrm{SL}_2(\mathbb{A}))$
- ℓ-adic representation of $\mathrm{Gal}(\bar{\mathbb{Q}}:\mathbb{Q})$.
- Euler product of degree 2 satisfying certain functional equations.
- Homology / cohomology class on a Riemann surface.
- Class in the group cohomology of an arithmetic group.

Pre-requisites.

- Complex analysis
- Real analysis
- Group theory
- Elementary number theory
- Some topology

0.2. Course plan

- Doubly periodic functions; holomorphism Eisenstein series; the space of lattices.
- Theta functions and q-series.
- Hyperbolic geometry and Fuchsian groups.
- Holomorphic forms and Fourier coefficients.
- Hecke operators.
- The Hecke L-series and Weil's converse theorem.
- Other topics.

References. These notes were constructed mainly using the books of Miyake and Diamond-Shurman. The chapter on Fuchsian groups is based on the book of Katok. Other references include:

- Iwaniec
- Iwaniec

- Iwaniec-Kowalski
- Shimura73

CHAPTER 1

Introduction

Math 613: Problem set 1 (due 15/9/09)

Some number theory

1. For a commutative ring R write $R^\times$ for the group of invertible elements, $\mathrm{GL}_n(R)$ for the group $\{g \in M_n(R) \mid \det g \in R^\times\}$, and $\mathrm{SL}_n(R)$ for $\{g \in M_n(R) \mid \det g = 1\}$.
   (a) Show that $\mathrm{GL}_n(\mathbb{Z})$, $\mathrm{GL}_n(\mathbb{Z}/N\mathbb{Z})$ are the automorphism groups of the additive groups of the rings $\mathbb{Z}^n$, $(\mathbb{Z}/N\mathbb{Z})^n$ respectively.
   OPT Show that $\mathrm{GL}_n(R)$ is the automorphism group of the R-module R^n.
   (b) Let N_1, N_2 be relatively prime and let $N = N_1N_2$. Show that $\mathrm{GL}_n(\mathbb{Z}/N\mathbb{Z}) \simeq \mathrm{GL}_n(\mathbb{Z}/N_1\mathbb{Z}) \times \mathrm{GL}_n(\mathbb{Z}/N_2\mathbb{Z})$.
   (c) Show that the maps $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ (reduction mod N) are surjective.
   Hint: Given $\bar{\gamma} \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ choose a pre-image $\gamma \in M_2(\mathbb{Z})$ such that the entries in the bottom row of γ are relatively prime.
   (d) Find the image of the map $\mathrm{GL}_2(\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$.
   Hint: What is $\mathbb{Z}^\times$?
   OPT Do parts (c),(d) for SL_n, GL_n.
   OPT Do parts (c),(d) replacing $\mathbb{Z}$ with the ring of integers of a number field and N with an ideal in the ring of integers.

2. Let G be a group, $H \operatorname{char} G$ a *characteristic* subgroup. In other words, one such that for every automorphism $\sigma \in \mathrm{Aut}(G)$ we have $\sigma(H) = H$.
   (a) Show $H \lhd G$.
   (b) Show that there is a natural map $\mathrm{Aut}(G) \to \mathrm{Aut}(G/H)$.
   *(c) Classify the orbits of $\mathrm{Aut}(\mathbb{Z}^2)$ on $\mathbb{Z}^2$.
   (d) Find all chracteristic subgroups of $\mathbb{Z}^2$.
   OPT Do parts (c),(d) in $\mathbb{Z}^n$.

Lattices in $\mathbb{R}^n$

3. (Construction) Let $\{v_1, \ldots, v_k\} \subset \mathbb{R}^n$ be linearly independent, let $\Lambda = \left\{\sum_{j=1}^k a_j v_j \mid \underline{a} \in \mathbb{Z}^k\right\} \subset \mathbb{R}^n$ be the subgroup they generate, and let $\mathbb{R}^n/\Lambda$ be the quotient group, endowed with the quotient topology coming from the map $\pi \colon \mathbb{R}^n \to \mathbb{R}^n/\Lambda$.
   (a) Show that the map $\mathbb{Z}^k \to \Lambda$ given by $\underline{a} \to \sum_j a_j v_j$ is an isomorphism.
   (b) Show that Λ is a discrete subset of $\mathbb{R}^n$.
   (c) Given $x, y \in \mathbb{R}^n$ such that $\pi(x) \neq \pi(y)$ find open sets $U_x, U_y \subset \mathbb{R}^n$ containing x, y respectively such that $\pi(U_x) \cap \pi(U_y) = \emptyset$. You have shown that $\mathbb{R}^n/\Lambda$ is Hausdorff.
   Hint: Let $r = \min\{\|v\| \mid v \in \Lambda, v \neq 0\}$.
   (d) Show that $\mathbb{R}^n/\Lambda$ isn't compact if $k < n$.
   (e) Let $k = n$, and let $\mathcal{F} = \left\{\sum_{j=1}^k a_j v_j \mid \forall j : |a_j| \leq \frac{1}{2}\right\}$. Show that $\mathcal{F}$ surjects onto $\mathbb{R}^n/\Lambda$ and conclude that $\mathbb{R}^n/\Lambda$ is compact.
   HINT Applying an automorphism of $\mathbb{R}^n$ before starting the problem will make your life much easier.

4. (Reduction theory) Let $\Lambda \subset \mathbb{R}^n$ be a discrete subgroup. Set $\Lambda_0 = \{0\}$, $V_0 = \{0\}$ and for $j \geq 1$ if $\Lambda \not\subset V_{j-1}$ choose $v_j \in \Lambda \setminus V_{j-1}$ minimizing the distance to V_{j-1}. Then set $\Lambda_j = \Lambda_{j-1} + \mathbb{Z}v_j$, $V_j = V_{j-1} + \mathbb{R}v_j$.
   (a) Assume by induction that $\Lambda_{j-1} = \Lambda \cap V_{j-1}$ and that it is a lattice in V_{j-1}. Show that set of distances $\left\{d(v, V_{j-1})\right\}_{v\in\Lambda}$ has a minimal non-zero member, so that v_j exists.
   Hint: Consider first the set of distances $d(v, V_{j-1})$ for vectors v whose orthogonal projection to V_{j-1} lies in $\mathcal{F}_{j-1} = \left\{\sum_{i=1}^{j-1} a_i v_i \mid |a_i| \leq \frac{1}{2}\right\}$.
   (b) Show that $\Lambda_j = \Lambda \cap V_j$.
   (c) Conclude that $\Lambda = \mathbb{Z}v_1 \oplus \cdots \mathbb{Z}v_j$ for some $0 \leq j \leq n$.

DEFINITION. Call $\Lambda < \mathbb{R}^n$ a *lattice* if it is discrete and if $\mathbb{R}^n/\Lambda$ is compact.

Convergence Lemma

Write $B(R)$ for the closed ball of radius R in $\mathbb{R}^n$, c_n for the volume of $B(1)$ so that $\mathrm{vol}(B(R)) = c_n R^n$. Fix a lattice $\Lambda < \mathbb{R}^n$.

5. Show that there exist $V, C > 0$ such that for any $R \geq 1$,
$$|\#(\Lambda \cap B(R)) - VR^n| \leq CR^{n-1}.$$
Hint: Consider the set $\bigcup_{v\in\Lambda\cap B(R)} (v + \mathcal{F})$, and prove the claim first for $R \geq 2\,\mathrm{diam}(\mathcal{F})$.

6. For $s \in \mathbb{C}$ the *Epstein zetafunction* is given by
$$E(\Lambda; s) = \sum_{v\in\Lambda}{}' \|v\|^{-ns},$$
where the prime indicates summation over non-zero elements of Λ.
   (a) Show that the series defining $E(\Lambda; \sigma)$ converges for any real $\sigma > 1$.
   Hint: You can use 5, or the identity $\int_{\mathbb{R}^n} f(x)dx = \sum_{v\in\Lambda} \int_{v+\mathcal{F}} f(x)dx$.
   (b) Show that the series defining $E(\Lambda; s)$ converges uniformly absolutely in any right half-plane of the form $\Re(s) \geq \sigma > 1$.
   (c) Conclude that the series defines a holomorphic function in the open half-plane $\Re(s) > 1$.
   (d) For $n = 1$ relate $E(\Lambda; s)$ to the Riemann zetafunction.

REMARK. In the next problem set we will analytically continue $E(\Lambda; s)$, showing that it extends to a meromorphic function on $\mathbb{C}$ bounded in vertical strips with poles at $0, 1$ and satisfying a functional equation relating the values at s and $1 - s$.

Later in the course we will also fix s and consider $E(\Lambda; s)$ as a function of Λ.

Extra: The "moduli space of complex annuli"

8. Given $0 < r < s$ let Let $A_{r,s} = \{z \in \mathbb{C} \mid r < |z| < s\}$. Write A_r for $A_{r,1}$. Show that $A_{r,s}$ and $A_{r',s'}$ are biholomorphic when $r'/s' = r/s$.

9. Let $f\colon A_r \to A_{r'}$ be a biholomorphism.
   (a) Show that as $z \to \partial A_r$, $f(z) \to \partial A_{r'}$.
   (b) Show that for $\varepsilon > 0$ and all small enough δ (depending on ε), $f(A_{r+\delta,1-\delta}) \supset A_{r'+\varepsilon,1-\varepsilon}$. Conclude that, up to inversion, we have $|f(z)| \xrightarrow[|z|\to 1]{} 1$ and $|f(z)| \xrightarrow[|z|\to r]{} r'$.
   (c) Let $g(z) = \log r \log|f(z)| - \log r' \log|z|$. Show that g is harmonic in A_r and vanishes at ∂A_r. Conclude that $g(z) = 0$.
   (d) Show that $f(z) = cz$ where $|c| = 1$, and hence that $r = r'$.

1.1. Eisenstein series from doubly periodic functions

1.1.1. The Weierstrass $\wp$-function. $F(z)=\int_0^z \frac{dx}{\sqrt{1-x^2}}$ is a complicated function, but its inverse $F^{-1}(z)=\sin z$ is actually quite easy to work with. It is an entire function and it is *periodic*. Trying to measure the arclength of an ellipse gives non-elementary integrals. *Abel* realized that these difficult integrals have nice inverses – the inverse functions are *doubly periodic functions*.

DEFINITION 1. A *lattice* in $\mathbb{R}^n$ is a cocompact discrete subgroup.

PROPOSITION 2. *(Problem set 1) Λ is lattice iff $\Lambda=\oplus_{i=1}^n \mathbb{Z}v_i$ for a basis.*

DEFINITION 3. A meromorphic function $f\colon \mathbb{C}\to\mathbb{C}$ is *doubly periodic* if there exist $\omega_1,\omega_2\in\mathbb{C}$, linearly independent over $\mathbb{R}$, such that $f(z)=f(z+\omega_1)=f(z+\omega_2)$. Equivalently, f is *doubly periodic* if there exists a lattice $\Lambda\subset\mathbb{C}$ such that $f(z+\omega)=f(z)$ for all $\omega\in\Lambda$.

EXERCISE 4. Let $\omega_1,\omega_2\in\mathbb{C}$ be linearly dependent over $\mathbb{Q}$. Show that there exists $\omega_3\in\mathbb{C}$such that $\mathbb{Z}\omega_1+\mathbb{Z}\omega_2=\mathbb{Z}\omega_3$.

Let $\omega_1,\omega_2\in\mathbb{C}$ be linearly dependent over $\mathbb{R}$ but linearly independent over $\mathbb{Q}$. Show that any meromorphic function which is invariant under $\mathbb{Z}\omega_1\oplus\mathbb{Z}\omega_2$ is constant.

Fix a lattice $\Lambda=\mathbb{Z}\omega_1\oplus\mathbb{Z}\omega_2\subset\mathbb{C}$. Let $\mathcal{F}$ be the *fundamental domain* $\{t\omega_1+sw_2 \mid 0\le t,s\le 1\}$.

EXERCISE 5. (properties of fundamental domains)

By Liouville's theorem any holomorphic doubly periodic function is constant.

LEMMA 6. *Let f be non-constant, Λ-periodic and meromorphic. Then f has the same number of zeroes and poles in $\mathcal{F}$, which is at least two.*

PROOF. $\frac{1}{2\pi i}\oint_{\partial\mathcal{F}}\frac{f'}{f}dz=0$ by periodicity so the same number of zeroes and poles. But $\frac{1}{2\pi i}\oint_{\partial\mathcal{F}}f(z)dz=0$ also, so can't have a unique simple pole. □

DEFINITION 7. the Weierstrass $\wp$-function is

$$\wp(z)=\frac{1}{z^2}+\sum_{\omega}{}'\left(\frac{1}{(z-\omega)^2}-\frac{1}{\omega^2}\right).$$

PROPOSITION 8. *The sum converges locally uniformly absolutely on $\mathbb{C}\setminus\Lambda$, and hence defines a holomorphic function there. This function is Λ-periodic and has double poles at the lattice points.*

PROOF. $\frac{1}{(z-\omega)^2}-\frac{1}{\omega^2}=O(\frac{1}{|\omega|^3})$ locally uniformly in $z\in\mathbb{C}\setminus\Lambda$ gives the convergence. This also shows that $\wp$ has a double pole at $z=0$. We can then differentiate term-by-term to see

$$\wp'(z)=-2\sum_{\omega\in\Lambda}\frac{1}{(z-\omega)^3}$$

which is evidently Λ-periodic. For $\omega\in\Lambda$ the derivative of $\wp(z+\omega)-\wp(z)$ then vanishes, so this function is constant. Since $\wp(\frac{1}{2}\omega)=\wp(-\frac{1}{2}\omega)$ (the function is even) the constant is zero. □

THEOREM 9. *The field of meromorphic Λ-periodic functions is precisely $\mathbb{C}(\wp,\wp')$.*

PROOF. Let f be meromorphic and Λ-periodic. We address first the case of f with poles only at lattice points, by induction on the order r of the pole at 0. If $r = 0$ f is constant. Otherwise we have $r \geq 2$; say $f(z) = \frac{A}{z^r} + O(z^{-r-1})$ near 0. Either r or $r-3$ is even; in either case there are $a, b \in \mathbb{Z}_{\geq 0}$ and B such that $B\wp(z)^a \wp'(z)^b = \frac{A}{z^r} + O(z^{-r-1})$ as well, at which point $f(z) - B\wp(z)^a\wp'(z)^b$ is Λ-periodic, meromorphic, has only poles at lattice points with order $< r$.

In general, let $\left\{\zeta_j\right\}_{j=0}^{r}$ be a set of representatives for the distinct poles of f in $\mathbb{C}/\Lambda$ with degrees d_j, where $\zeta_0 = 0$. Consider the function

$$g(z) = f(z) \prod_{j=1}^{r} \left(\wp(z) - \wp(\zeta_j)\right)^{d_j} .$$

It is Λ-periodic, meromorphic, and is regular at each ζ_j, hence an element of $\mathbb{C}(\wp.\wp')$ and we are done. □

DEFINITION 10. For $k > 2$ the *Eisenstein series* is the functions

$$G_k(\Lambda) = \sum_{\omega\in\Lambda}{}' \frac{1}{\omega^k} .$$

the convergence is verified in Problem Set 1.

Note that $G_k(\Lambda) = 0$ for odd k.

PROPOSITION 11. *Near* 0 *we have*

$$\wp(z) = \frac{1}{z^2} + \sum_{\substack{k=2 \\ \text{even}}}^{\infty} (k+1) G_k(\Lambda) z^k$$

PROOF. If $|z| < |\omega|$ we have

$$\frac{1}{\omega - z} = \omega^{-1} \frac{1}{1 - \frac{z}{\omega}} = \omega^{-1} \sum_{k=0}^{\infty} \left(\frac{z}{\omega}\right)^k .$$

Differentiating we find

$$\frac{1}{(z-\omega)^2} = \sum_{k=0}^{\infty} (k+1)\omega^{-k-2} z^k .$$

Summing over $\omega \in \Lambda$ we find

$$\sum_{\omega\in\Lambda}{}' \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right) = \sum_{k=1}^{\infty} (k+1) G_{k+2}(\Lambda) z^k .$$

□

LEMMA 12. *Let* $\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$.

(1) $\wp(z)$ *takes every complex value twice on* $\mathcal{F}$. *More precisely, if* $\wp(z) = \wp(w)$ *then* $w \equiv \pm z\,(\Lambda)$.
(2) *The zeroes of* $\wp'(z)$ *on* $\mathcal{F}$ *are at* $\frac{1}{2}\Lambda \setminus \Lambda$. *Representatives in the fundamental domain are* $z_1 = \omega_1/2$, $z_2 = \omega_2/2$ *and* $z_3 = (\omega_1 + \omega_2)/2$.

PROOF. $\frac{1}{2\pi i}\oint_{\partial\mathcal{F}}\frac{f'(z)}{f(z)-\zeta}dz=0$ by periodicity, so f takes every value twice counting multiplicity. This completes part (1)

Since $\wp'(z)$ is periodic and has a triple pole at $z=0$ it has exactly three zeroes in $\mathcal{F}$ (counting multiplicity). Since $z_i-(-z_i)=2z_i\in\Lambda$ we have $\wp'(z_i)=\wp'(z_i)$ by periodicity and $\wp'(z_i)=-\wp'(z_i)$ since the function is odd. □

THEOREM 13. *($\mathbb{C}/\Lambda$ as an elliptic curve)*

(1) *Let $g_2(\Lambda)=60G_4(\Lambda)$, $g_3(\Lambda)=140G_6(\Lambda)$. Then*

$$(\wp'(z))^2=4(\wp(z))^3-g_2(\Lambda)\wp(z)-g_3(\Lambda)\,.$$

(2) *The map $\mathbb{C}\to\hat{\mathbb{C}}\times\hat{\mathbb{C}}$ mapping $z\mapsto(\wp(z),\wp'(z))$ induces a bijection between $\mathbb{C}/\Lambda$ and the set of (projective) solutions to*

$$y^2=4x^3-g_2x-g_3\,.$$

(3) *The polynomial $4x^3-g_2x-g_3$ has three distinct roots $e_i=\wp_i(z_i)$.*

PROOF. Using the explicit Laurent expansion of Proposition 11 on sees that

$$4(\wp(z))^3-g_2(\Lambda)\wp(z)-g_3(\Lambda)-(\wp'(z))^2=O(z^2)$$

near $z=0$. The function on the left is Λ-periodic and cannot have poles except at lattice points. Since the singularity at $z=0$ is removable this is an entire function, hence constant. The asymptotics as $z\to0$ show that the constant is zero. This shows that the map of (2) is well-defined. That it is bijective is Lemma 12, which also shows that the e_i are distinct. □

In other words, the transcendental map $z\mapsto(\wp(z),\wp'(z))$ gives $\mathbb{C}/\Lambda$ the structure of a complex algebraic variety, specifically a projective curve. It turns out there can be very few maps between elliptic curves.

COROLLARY 14. *The map $\Delta(\Lambda)=g_2^3-27g_3^2$ is nowhere vanishing on the space of lattices. The map $j(\Lambda)=1728\frac{g_2^3}{\Delta}$ is thus well-defined.*

THEOREM 15. *Let Λ,Λ' be lattices in $\mathbb{C}$ and let $f\colon\mathbb{C}/\Lambda\to\mathbb{C}/\Lambda'$ be holomorphic. Then there exist $a,b\in\mathbb{C}$ with $a\Lambda\subset\Lambda'$ such that $f(z+\Lambda)=az+b+\Lambda'$.*

PROOF. Let $\pi\colon\mathbb{C}\to\mathbb{C}/\Lambda$ be the quotient map. Then $f\circ\pi\colon\mathbb{C}\to\mathbb{C}/\Lambda'$ is continuous. Let $b\in\mathbb{C}$ be such that $f(0+\Lambda)=b+\Lambda'$. Since $\mathbb{C}$ is simply connected and π' is a universal covering map there exists a unique $\tilde{f}\colon\mathbb{C}\to\mathbb{C}$ such that $f\circ\pi=\pi'\circ\tilde{f}$ and $\tilde{f}(0)=b$. Differentiating in the obviuos co-ordinates one sees that $\left\|\tilde{f}'\right\|_\infty\le\|f'\|_\infty<\infty$ so $\tilde{f}$ is a holomorphic function of at most linear growth. It follows that there exists $a\in\mathbb{C}$ such that $\tilde{f}(z)=az+b$. That $a\Lambda\subset\Lambda'$ is clear. □

COROLLARY 16. *The elliptic curves $\mathbb{C}/\Lambda$ and $\mathbb{C}/\Lambda'$ are isomorphic iff $\Lambda=a\Lambda'$ for some $a\in\mathbb{C}^\times$. In that case $j(\Lambda)=j(\Lambda')$. We will later see that the j-invariant classifies elliptic curves.*

1.1.2. Modulfar forms on the space of elliptic curves. We now consider the situation as Λ varies. Let $\tilde{\mathcal{L}}_2$ denote the space of lattices in $\mathbb{C}$.

DEFINITION 17. (Provisional) Say that $f\colon\tilde{\mathcal{L}}_2\to\mathbb{C}$ has *weight k* if $f(r\Lambda)=r^{-k}f(\Lambda)$.

Let $Y(1) = \tilde{\mathcal{L}}_2/\mathbb{C}^\times$ be the space of lattices up to scaling, that is the space of isomorphism classes of elliptic curves. To understand $Y(1)$ we consider its universal cover, the space $\mathbb{H}$ of unordered bases up to scaling. Rotating and scaling every element of $\mathbb{H}$ has unique representative of the form $\{1,\tau\}$ with $\Im(\tau) > 0$, associated to the lattice $\Lambda_\tau = \mathbb{Z} \oplus \mathbb{Z}\tau$.

We would now like to understand the isomorphism relation in this parametrization.

LEMMA 18. $\mathbb{C}/\Lambda_\tau \simeq \mathbb{C}/\Lambda_{\tau'}$ *iff* $\tau' = \frac{a\tau+b}{c\tau+d}$ *for some* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.

PROOF. Say $r\Lambda_{\tau'} = \Lambda_\tau$. Then there exist $a,b,c,d \in \mathbb{Z}$ such that $r\tau' = a\tau + b$ and $r = c\tau + d$. It follows that $\tau' = \frac{a\tau+b}{c\tau+d}$. A short calculation shows that $\frac{\tau}{r} = a'\tau' + b'$ and $\frac{1}{r} = c'\tau' + d'$ where a',b',c',d' are the entries of the inverse matrix. By uniqueness of representation it follows that $a',b',c',d' \in \mathbb{Z}$ so $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$. Now

$$\Im\left(\frac{a\tau+b}{c\tau+d}\right) = \Im\left(\frac{(a\tau+b)(c\bar{\tau}+d)}{|c\tau+d|^2}\right) = \frac{ad-bc}{|c\tau+d|^2}\Im(\tau)$$

so if $\Im(\tau), \Im(\tau') > 0$ we have $ad - bc > 0$ so $ad - bc = 1$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. The converse is clear. $\square$

LEMMA 19. *The map* $(\gamma,\tau) \mapsto \frac{a\tau+b}{c\tau+d}$ *is an action of* $\mathrm{SL}_2(\mathbb{Z})$ *on* HH.

COROLLARY 20. $Y(1) = \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$.

The analysis has boiled down to:

PROPOSITION 21. $G_k(\tau) \overset{def}{=} G_k(\Lambda_\tau)$ *is a holomorphic function on* $\mathbb{H}$ *satisfying*

$$G_k(\gamma\tau) = (cz+d)^k G_k(\tau)\,.$$

DEFINITION 22. An *automorphic form* or *weak modular form* of *weight k* on $Y(1)$ is a function satisfying these two properties. The space of these will be denoted $\mathcal{A}_k(\mathrm{SL}_2(\mathbb{Z}))$. Write $\mathcal{A}(\mathrm{SL}_2(\mathbb{Z}))$ for the direct sum, a graded algebra.

REMARK 23. Note that G_k is not a function on $Y(1)$, since it is not $\mathrm{SL}_2(\mathbb{Z})$-invariant. It is a section of a complex line bundle on that sufrace, however.

We now show that G_k is also a generating function. For motivation note that $\Lambda_{\tau+1} = \Lambda_\tau$ so $G_k(\tau+1) = G_k(\tau)$, so the restriction of G_k to any line of the form $\mathbb{R} + iy$ is a periodic function. The co-ordinate $q = e(\tau)$ is a biholmorphism between the cylinder $\Gamma_\infty\backslash\mathbb{H}$ and the punctured disc $\mathbb{D}\setminus\{0\}$. Since $G_k(q)$ is holomorphic in the disc, we have a Laurent expansion

$$G_k(\tau) = \sum_{m\in\mathbb{Z}} a_m q^m\,.$$

DEFINITION 24. Call a_m the *Fourier coefficients* of G_k.

REMARK 25. To obtain this expansion in an analytic way restrict the function to a line $\mathbb{R} + iy$. It is a smooth periodic function, and hence

$$G_k(\tau) = \sum_{m\in\mathbb{Z}} a_m(y) e(mx)$$

for some function $a_m(y)$. Then, integrating the Cauchy-Riemann equation against $e(-mx)$ on $[0,1]+iy$ shows that $a_m(y) = a_m e(miy)$.

DEFINITION 26. Let f be a *weak modular form* of weight k for $\mathrm{SL}_2(\mathbb{Z})$ with Fourier expansion $f = \sum_m a_m q^m$. We say that f is a:

(1) *Meromorphic* form if $a_f(m) = 0$ for $m < M_0$ (the space of these is denoted $\mathcal{A}_k(\mathrm{SL}_2(\mathbb{Z}))$;
(2) *Holomorphic* form *if* $a_f(m) = 0$ *for* $m < 0$ (the space of these is denoted $\mathcal{M}_k(\mathrm{SL}_2(\mathbb{Z}))$;
(3) *Cusp* form *if* $a_f(m) = 0$ *for* $m < 1$ (in this case say that f is *normalized* if $a_1 = 1$) (the space of these is denoted $\mathcal{S}_k(\Gamma)$

REMARK 27. $\mathcal{S}(\Gamma) = \oplus_k \mathcal{S}_k(\Gamma)$ is an ideal in $\mathcal{M}(\Gamma) = \oplus_k \mathcal{M}_k(\Gamma)$.

For $k \in \mathbb{R}$ write $\sigma_k(n) = \sum_{d|n} d^k$ for the divisor function. We now show that E_k is the generating function for $\sigma_{k-1}(n)$.

PROPOSITION 28. *For $k \geq 4$ we have*

$$G_k(\tau) = 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n .$$

PROOF. Since $\Lambda_\tau = \{c\tau + d \mid c,d \in \mathbb{Z}\}$ and since k is even we have

$$G_k(\tau) = \sum_{d\in\mathbb{Z}}\frac{1}{d^k} + 2\sum_{c>0}\sum_{d\in\mathbb{Z}}\frac{1}{(c\tau+d)^k} .$$

In problem set 2 it wil be established that $\sum_{d\in\mathbb{Z}}\frac{1}{(\tau+d)^k} = \frac{(-2\pi i)^k}{(k-1)!}\sum_{m=1}^{\infty} m^{k-1}e(m\tau)$. Since all sums are absolutely convergent it follows that

$$G_k(\tau) = 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!}\sum_{c>0}\sum_{m=1}^{\infty} m^{k-1}e(mc\tau) ,$$

□

COROLLARY 29. *The Fourier expansion of $\frac{\Delta(\Lambda_\tau)}{(2\pi)^{12}}$ begins $q + \cdots$ with integral coefficients. The Fourier expansion of $j\colon \mathbb{H} \to \mathbb{C}$ begins $\frac{1}{q} + 744 + \cdots$ (all integers).*

REMARK 30. We will later see $j\colon Y(1) \to \mathbb{C}$ is bijective; "filling in the cup" gives a bijection $X(1) \to \hat{\mathbb{C}} = \mathbb{P}^1(\mathbb{C})$, realizing the Riemann surface $X(1)$ as a complex algebraic variety. In fact, $\mathbb{C}(j)$ is the function field of $X(1)$ since

$$\frac{\prod_a (j(\tau) - j(\tau_a))}{\prod_b (j(\tau) - j(\tau_b))}$$

can match the finite divisor of any g (hence the divisor since the total degree is zero), and any function without poles on $X(1)$ is constant.

DEFINITION 31. The *normalized* Eisenstein series is

$$\begin{aligned} E_k(\tau) &= \frac{1}{2\zeta(k)}G_k(\tau) \\ &= 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n \end{aligned}$$

where B_k is the kth Bernoulli number. In problem set 2 it will be verified that $B_k \in \mathbb{Q}$ so that $E_k(\tau)$ has rational coefficients with a common denominator.

EXAMPLE 32. Consider the two elements $E_8(\tau), (E_4(\tau))^2 \in \mathcal{M}_8(Y(1))$. A calculation in complex algebraic geometry shows $\dim_{\mathbb{C}} \mathcal{M}_8(Y(1)) = 1$. Comparing the constant coefficients both functions are equal. It follows that

$$\sigma_7(n) = \sigma_3(n) + 120 \sum_{j=1}^{n-1} \sigma_3(j)\sigma_3(n-j).$$

REMARK 33. The same calculation shows that (note the order of summation!)

$$\begin{aligned} G_2(\tau) &\stackrel{\text{def}}{=} \sum_{d \in \mathbb{Z}} \frac{1}{d^2} + 2 \sum_{c>0} \sum_{d \in \mathbb{Z}} \frac{1}{(c\tau+d)^2} \\ &= 2\zeta(2) - 8\pi^2 \sum_{n=1}^{\infty} \sigma(n) q^n. \end{aligned}$$

It follows that

$$\begin{aligned} \tau^{-2} G(-1/\tau) &= \sum_{c \in \mathbb{Z}} \sum_{d \in \mathbb{Z}_c} \frac{1}{(d\tau - c)^2} \\ &= 2\zeta(2) + \sum_{d \in \mathbb{Z}} \sum_{c \neq 0} \frac{1}{(c\tau+d)^2}. \end{aligned}$$

Also,

$$\begin{aligned} G_2(\tau) &= G_2(\tau) - \sum_{c \neq 0} \sum_{d \in \mathbb{Z}} \frac{1}{(c\tau+d)(c\tau+d+1)} \\ &= 2\zeta(2) + \sum_{c \neq 0} \sum_{d \in \mathbb{Z}} \frac{1}{(c\tau+d)^2(c\tau+d+1)} \\ &= 2\zeta(2) + \sum_{d \in \mathbb{Z}} \sum_{c \neq 0} \frac{1}{(c\tau+d)^2(c\tau+d+1)} \end{aligned}$$

which converges absolutely. Subtracting we find

$$
\begin{aligned}
G_2(\tau)-\tau^{-2}G_2(-1/\tau) &= -\sum_{d\in\mathbb{Z}}\sum_{c\neq 0}\frac{1}{(c\tau+d)(c\tau+d+1)}\\
&= -\lim_{N\to\infty}\sum_{d=-N}^{N-1}\sum_{c\neq 0}\left(\frac{1}{c\tau+d}-\frac{1}{c\tau+d+1}\right)\\
&= -\lim_{N\to\infty}\sum_{c\neq 0}\left(\frac{1}{c\tau-N}-\frac{1}{c\tau+N}\right)\\
&= -\frac{2}{\tau}\lim_{N\to\infty}\sum_{c=1}^{\infty}\left(\frac{1}{-N/\tau+c}+\frac{1}{-N/\tau-c}\right)\\
&= -\frac{2}{\tau}\lim_{N\to\infty}\left(\frac{\tau}{N}-\pi i-2\pi i\sum_{m=1}^{\infty}e(-2\pi imN/\tau)\right)\\
&= \frac{2\pi i}{\tau}.
\end{aligned}
$$

It follows that

$$(c\tau+d)^{-2}G_2(\gamma\tau)=G_2(\tau)-\frac{2\pi ic}{c\tau+d}$$

for all $\gamma\in\mathrm{SL}_2(\mathbb{Z})$.

1.2. The modular discriminant and partitions

DEFINITION 34. The Dedekind etafunction is (roughly, the inverse of the partition function)

$$\eta(\tau)=q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)$$

PROPOSITION 35. $\eta(-1/\tau)=\sqrt{-i\tau}\eta(\tau)$.

PROOF. Let $E_2(\tau)=\frac{1}{2\zeta(2)}G_2(\tau)=1-24\sum_{n=1}^{\infty}\sigma(n)q^n$. Since $\log(\eta(\tau))=\frac{\pi i}{12}\tau+\sum_{n=1}^{\infty}\log(1-q^n)$ converges localy uniformly absolutely it can be differentiated termwise, giving

$$
\begin{aligned}
(\log(\eta(\tau))' &= \frac{\pi i}{12}-2\pi i\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\\
&= \frac{\pi i}{12}-2\pi i\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}nq^{nm}\\
&= \frac{\pi i}{12}-2\pi i\sum_{d=1}^{\infty}\sigma(d)q^d\\
&= \frac{\pi i}{12}E_2(\tau).
\end{aligned}
$$

Thus

$$\frac{d}{d\tau}\log(\eta(-\frac{1}{\tau}))=\frac{1}{\tau^2}\frac{\pi i}{12}E_2(-\frac{1}{\tau})$$

and
$$\frac{d}{d\tau}\log\left(\sqrt{-i\tau}\eta(\tau)\right)=\frac{1}{2\tau}+\frac{\pi i}{12}E_2(\tau)=\frac{\pi i}{12}\left(E_2(\tau)+\frac{12}{2\pi i\tau}\right).$$
It follows that $\eta(-1/\tau)=C\sqrt{-i\tau}\eta(\tau)$ for some C. Set $\tau=-i$ to see $C=1$. □

DEFINITION 36. The Ramanujan Δ-function and τ-function are given by:
$$\begin{aligned}\Delta(\tau) &\overset{\text{def}}{=} q\prod_{n=1}^{\infty}(1-q^n)^{24}\\ &\overset{\text{def}}{=} \sum_{n=1}^{\infty}\tau(n)q^n.\end{aligned}$$

COROLLARY 37. $\Delta(\tau)\in S_{12}(Y(1))$.

CONJECTURE 38. *(Ramanujan)*

(1) $\tau(n)\tau(m)=\sum_{d|}\tau()$
(2) $|\tau(p)|\leq 2p^{\frac{11}{2}}$.

FACT 39. $\dim_{\mathbb{C}} S_{12}(1)=1$.

COROLLARY 40. $\Delta(\Lambda_\tau)=(2\pi)^{12}\Delta(\tau)$ *(examine a_0).*

THEOREM 41. *(Mordell 1923) Conjecture (1) holds.*

PROOF. For $f\in S_k(1)$ let $T_nf=\sum_{ad=n}\sum_{b(d)}d^{-k}f\left(\frac{az+b}{d}\right)$ ("nth Hecke operator"). Then T_nf are linear operators on $S_k(1)$ and it is easy to check that $T_nT_m=\sum_d T$. If $T_n\Delta=\lambda(n)\Delta$ then $\lambda_f(n)\lambda_f(m)$ satisfy Ramanujan's relation. It is also easy to check that if $f\in S_k(1)$ has the Fourier expansion $\sum_{n=1}^{\infty}a_f(n)q^n$ and $T_nf=\lambda_f(n)f$ then $a_f(n)=\lambda_f(n)a_f(1)$. In particular it follows that $\tau(n)=\lambda(n)$ and we are done. □

THEOREM 42. *(Deligne 1974) Conjecture (2) holds.*

This is deep. The main ingredient are the Weil conjectures.

CONJECTURE 43. *(Lehmer) $\tau(p)\neq 0$ for all p.*

1.3. Theta functions from sums of squares

Let $r_k(n)=\#\left\{\underline{v}\in\mathbb{Z}^k \mid \sum_{i=1}^k v_i^2=n\right\}$. Let $\theta_k(q)=\sum_{n=0}^{\infty}r_k(n)q^n$. Note that $\theta_k(q)=(\theta(q))^k$ for $\theta=\theta_1$.

Consider θ as a fuction of τ. Clearly $\theta(\tau+1)=\theta(\tau)$. Also,
$$\theta(\tau)=\sqrt{\frac{1}{-2i\tau}}\theta(-1/4\tau)$$
by Poisson summation. Since $\begin{pmatrix}1 & \\ 4 & 1\end{pmatrix}=\begin{pmatrix} & 1/4\\ -1 & \end{pmatrix}\begin{pmatrix}1 & -1\\ & 1\end{pmatrix}\begin{pmatrix} & -1\\ 4 & \end{pmatrix}$ one can apply the tranformation rules to find out:
$$\theta\left(\frac{\tau}{4\tau+1}\right)=\sqrt{4\tau+1}\theta(\tau),$$
That is $\theta\in\mathcal{M}_{\frac{1}{2}}(\Gamma)$ where $\Gamma_\theta<\mathrm{SL}_2(\mathbb{Z})$ is the subgroup generated by $\left\{\pm\begin{pmatrix}1 & \\ 4 & 1\end{pmatrix},\pm\begin{pmatrix}1 & 1\\ & 1\end{pmatrix}\right\}$.

COROLLARY 44. $\theta^4 \in \mathcal{M}_2(\Gamma_\theta)$.

Now verify $G_{2,N} = G_2(\tau) - NG_2(N\tau) \in \mathcal{M}_2(\Gamma_\theta)$ for $N = 2, 4$.

FACT 45. $\dim_{\mathbb{C}} \mathcal{M}_2(\Gamma_\theta) = 2$.

COROLLARY 46. $\theta^4 = aG_{2,2} + bG_{2,4}$ *for some* A, B.

Examining the first two Fourier coefficents ($\theta^4 = 1 + 8q + \cdots$) shows:

$$\theta^4(\tau) = -\frac{1}{\pi^2} G_{2,4}(\tau) = 1 + 8 \sum_{n=1}^{\infty} \left(\sum_{\substack{d|n \\ 4\nmid d}} d \right) q^n .$$

THEOREM 47. *(Jacobi) For* $n \geq 1$ *we have* $r_4(n) = \sum_{\substack{d|n \\ 4\nmid d}} d$.

In general we have $\theta^{4k} \in \mathcal{M}_{2k}(\Gamma_\theta)$.

FACT 48. $\mathcal{M}_{2k}(\Gamma_\theta)$ *is generated by 2 Eisenstein series and some cusp forms.*

COROLLARY 49. *Bounds on Fourier coefficients of cusp forms give asymptotics for* $r_{4k}(n)$.

Math 613: Problem set 2 (due 22/9/09)

Let V be an n-dimensional inner product space, and fix a lattice $\Lambda < \mathbb{R}^n$ with basis $\{v_j\}_{j=1}^n$. Write $\mathbb{T} = V/\Lambda$ for the quotient torus, a compact space.

Integration on $\mathbb{T}$

DEFINITION. A *fundamental domain* for Λ is a closed subset $\mathcal{F} \subset V$ such that:

(1) $\cup_{v\in\Lambda}(v+\mathcal{F}) = V$, that is $\mathcal{F}$ intersects every orbit and surjects on $\mathbb{T}$.
(2) There is an open set $\mathcal{F}^\circ$ which injects into $\mathbb{T}$ and such that $\mathcal{F} = \overline{\mathcal{F}^\circ}$.
(3) The difference set $\mathcal{F} \setminus \mathcal{F}^\circ$ has measure zero.

1. Show that $\mathcal{F}_{\frac{1}{2}} = \left\{\sum_{j=1}^n a_j v_j \mid |a_j| \le \frac{1}{2}\right\}$ and $\mathcal{F}_1 = \left\{\sum_{j=1}^n a_j v_j \mid 0 \le a_j \le 1\right\}$ are fundamental domains.

*2. (The Dirichlet domain) Fix $x_0 \in V$ and set

$$\mathcal{F}_{\mathrm{D}} = \{x \in V \mid \forall v \in \Lambda : \|x - x_0\| \le \|x - (x_0 + v)\|\} .$$

(a) Show that $\mathcal{F}_{\mathrm{D}}$ is closed and surjects on $\mathbb{T}$.
Hint: Write it as an intersection of closed half-spaces.
(b) Show that $\mathcal{F}_{\mathrm{D}}$ is bounded.
Hint: Show that $\mathcal{F}_{\mathrm{D}} \subset B(x_0, 2\,\mathrm{diam}(\mathcal{F}_1))$.
(c) Show that $\mathcal{F}_{\mathrm{D}}$ is the intersection of finitely many closed half-spaces.
(d) Let $\mathcal{F}_{\mathrm{D}}^\circ$ be the intersection of the interiors of these half-spaces and show that $\mathcal{F}_{\mathrm{D}}$ is a fundamental domain.

3. (Lattice averaging) A function $f \in C(V)$ is said to be *of rapid decay* if for all $N \ge 1$ the function $(1 + \|x\|)^N f(x)$ is bounded. $f \in C^\infty(V)$ is said to be of *Schwartz class* if it and all its derivatives are of rapid decay (the set of such functions is denoted $\mathcal{S}(V)$).

(a) Let f be of rapid decay. Show that for all $x \in V$, $(\Pi_\Lambda f)(x) \stackrel{\mathrm{def}}{=} \sum_{v\in\Lambda} f(x+v)$ converges and defines a continuous function on $\mathbb{T}$.
OPT Let $f \in \mathcal{S}(V)$. Show that $\Pi_\Lambda f$ is smooth.
(b) (Smooth fundamental domain) Let $\chi_0 \in C_{\mathrm{c}}^\infty(V)$ be non-negative and satisfy $\chi_0 \restriction_{\mathcal{F}} \equiv 1$ for some compact fundamental domain $\mathcal{F}$. Show that $\chi(x) = \frac{\chi_0(x)}{(\Pi_\Lambda \chi_0)(x)} \in C_{\mathrm{c}}^\infty(V)$ and that we have $\Pi_\Lambda \chi \equiv 1$.

*4. (Integration on $\mathbb{T}$) dx will denote the Lebesgue measure on V. For $f \in C(\mathbb{T})$ define $\int_{\mathbb{T}} f(\bar{x})d\bar{x} = \int_V f(x)\chi(x)dx$.

(a) Show that the integral on the RHS defines a linear map $C(\mathbb{T}) \to \mathbb{C}$ mapping non-negative functions to non-negative reals.
(b) Show that for any fundamental domain $\mathcal{F}'$ for Λ we have

$$\int_{\mathcal{F}'} f(x)dx = \int_V f\chi .$$

(c) Conclude that the measure $d\bar{x}$ on $\mathbb{T}$ is translation-invariant.

(d) Show that the volume of $\mathbb{T}$ is the absolute value of the determinant of the matrix A such that a_{ij} is the ith co-ordinate of v_j in an orthonormal basis of V. Conclude that $(\mathrm{vol}(\mathbb{T}))^2$ is the determinant of the *Gram matrix*, whose ijth entry is $\langle v_i, v_j \rangle$.
Hint: Use one of the fundamental domains of problem 1.

5. Let V be an n-dimensional real vector space, V^* the dual space. Let $\Lambda < V$ be a lattice, and set $\Lambda^* = \{v^* \in V^* \mid v^*(\Lambda) \subset \mathbb{Z}\}$.
(a) Show that Λ^* is a lattice in V^*.
Hint: Use the dual basis.
(b) Show that the standard isomorphism $V \simeq V^{**}$ identifies Λ with Λ^{**}.
(*c) If V is an inner product space show that $\mathrm{vol}(V/\Lambda)\,\mathrm{vol}(V^*/\Lambda^*) = 1$.

The Poisson Summation Formula

We will use the standard notation $e(z) = \exp(2\pi i z)$. For a short note on Fourier series and the Poisson Summation Formula see the course website.

6. Fix $k \geq 2$.
(a) Show that $\tau \mapsto \sum_{d\in\mathbb{Z}} \frac{1}{(\tau+d)^k}$ is holomorphic in $\mathbb{H} = \{x+iy \mid y > 0\}$.
(b) Show that
$$\int_{\mathbb{R}} \frac{e(-rx)}{(x+\tau)^k} dx = \begin{cases} \frac{(-2\pi i)^k}{(k-1)!} r^{k-1} e(r\tau) & r \geq 0 \\ 0 & r \leq 0 \end{cases}$$
(c) Show that $\sum_{d\in\mathbb{Z}} \frac{1}{(\tau+d)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{m=1}^{\infty} m^{k-1} e(m\tau)$.
(d) Show that there exists C such that
$$\frac{1}{\tau} + \sum_{d\geq 1} \left(\frac{1}{\tau+d} + \frac{1}{\tau-d} \right) = C + (-2\pi i) \sum_{m=0}^{\infty} e(m\tau) = C - \frac{2\pi i}{1-e(\tau)}.$$
Hint: After showing that both sides are differentiable, take their derivatives.
(e) Multiply by τ and use the Taylor expansion of both sides to show that $C = \pi i$ and that
$$1 - 2\sum_{k=1}^{\infty} \zeta(2k)\tau^{2k} = \pi i \tau \frac{e(\tau)+1}{e(\tau)-1}.$$
(f) Show that for $k \geq 1$ even, $\zeta(k) = -\frac{1}{2}\frac{(2\pi i)^k}{k!} B_k$ where B_k are rational numbers.
Hint: let $\frac{t}{2}\frac{e^t+1}{e^t-1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} t^k$.

7. let $\varphi(x) = \exp\left\{-\pi\alpha \|x\|^2\right\}$ on $\mathbb{R}^n$ where $\Re(\alpha) > 0$.
(a) Show that $\hat{\varphi}(k) = \alpha^{-n/2} \exp\left\{-\frac{\pi}{\alpha}\|k\|^2\right\}$ (take the branch of the square root defined on $\Re(\alpha) > 0$ such that $\sqrt{1} = 1$).
(b) Conclude that $\theta(\tau) = \sum_{n\in\mathbb{Z}} e^{2\pi i n^2 \tau}$ satisfies $\theta(-\frac{1}{4\tau}) = \sqrt{-2i\tau}\,\theta(\tau)$

Continuing the Epstein Zetafunction

8. For $\varphi \in C^\infty(V)$ of Schwartz class set $\varphi(\Lambda) = \sum'_{v\in\Lambda} \varphi(v)$ and $Z(\Lambda;\varphi;s) = \int_0^\infty \varphi(r\Lambda) r^{ns} \frac{dr}{r}$.
   (a) Show that the sum converges absolutely.
   (b) Show that as $r \to \infty$, $|\varphi|(r\Lambda)$ decays faster than any polynomial and that as $r \to 0$, $|\varphi|(r\Lambda) = O(r^{-n})$. Conclude that $Z(\Lambda;\varphi;s)$ converges absolutely in $\Re(s) > 1$ and defines a holomorphic function there.
   (c) Applying the Poisson summation formula, show that for $\Re(s) > 1$,
$$Z(\Lambda;\varphi;s) = \int_1^\infty \varphi(r\Lambda) r^{ns}\frac{dr}{r} - \varphi(0)\frac{1}{ns} + \frac{1}{\mathrm{vol}(\Lambda)}\int_1^\infty \hat{\varphi}(r\Lambda^*) r^{n(1-s)}\frac{dr}{r} - \frac{\hat{\varphi}(0)}{\mathrm{vol}(\Lambda)}\frac{1}{n(1-s)}.$$
   (d) Since $\varphi \in \mathcal{S}(V)$ we also have $\hat{\varphi} \in \mathcal{S}(V)$ and $\hat{\hat{\varphi}}(x) = \varphi(-x)$. Conclude that $Z(\Lambda;\varphi;s)$ extends to a meromorphic function of s with poles at $s = 0,1$ which satisfies the *functional equation*
$$\sqrt{\mathrm{vol}(\Lambda)}Z(\Lambda;\varphi;s) = \sqrt{\mathrm{vol}(\Lambda^*)}Z(\Lambda^*;\hat{\varphi};1-s).$$
   (e) Assume that φ is spherical, and show that for $\Re(s) > 1$ we have
$$Z(\Lambda;\varphi;s) = \left(\int_0^\infty \varphi(r) r^{ns}\frac{dr}{r}\right) E(\Lambda;s).$$
   (f) For $\varphi \in C_c^\infty(\mathbb{R}^\times_{>0})$, show that $\left(\int_0^\infty \varphi(r) r^{ns}\frac{dr}{r}\right)$ extends to an entire function; conclude that $E(\Lambda;s)$ extends to a meromorphic function of s.
   (g) For $\varphi(x) = \exp\left\{-\pi\|x\|^2\right\}$ show that $\varphi(x) = \hat{\varphi}(x)$ and that
$$\int_0^\infty \varphi(r) r^{ns}\frac{dr}{r} = 2\pi^{-ns/2}\Gamma(\frac{ns}{2}).$$
   Conclude that $E(\Lambda;s)$ has no poles other than $s = 0,1$ and satisfies the functional equation
$$\sqrt{\mathrm{vol}(\Lambda)}\pi^{-\frac{n}{2}\left(\frac{1}{2}+s\right)}\Gamma(\frac{ns}{2})E(\Lambda;s) = \sqrt{\mathrm{vol}(\Lambda^*)}\pi^{-\frac{n}{2}\left(\frac{1}{2}-s\right)}\Gamma(\frac{n(1-s)}{2})E(\Lambda^*;1-s).$$
   (h) Show from (c) that $\mathrm{Res}_{s=1} E(\Lambda;s) = \frac{\pi^{n/2}}{2\,\mathrm{vol}(\Lambda)\Gamma\left(\frac{n}{2}\right)}$.

REMARK 50. We often write $\mathrm{vol}(\Lambda)$ for the *covolume* $\mathrm{vol}(V/\Lambda)$.

CHAPTER 2

Fuchsian groups

2.1. The hyperbolic plane

$\mathbb{H}$ – upper half-plane model; metric, measure.

Isometries; NA acts simply transitively so $G = \mathrm{Isom}(\mathbb{H}) = NAK$, $\mathrm{Isom}^+(\mathbb{H}) = \mathrm{Isom}^\circ(\mathbb{H}) = NAK^\circ$. $P = NAM$.

Classification of isometries $\iff$ conjugacy classes in isometry group.

More geometry: Geodesics, boundary, classification revisited (fixed points). Gauss-Bonnet.

Math 613: Problem set 3 (due 4/10/09)

For a group G acting on a space X write $G\backslash X$ for the space of orbits. If X is a topological space, G a topological group and the action $G \times X \to X$ is continuous we endow $G\backslash X$ with the quotient topology.

The moduli space of elliptic curves

1. Let $T = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$, $S = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$ and let $\Gamma < \mathrm{SL}_2(\mathbb{Z})$ be the subgroup they generate, Γ_∞ the subgroup generated by $\pm T$ (note that $-I$ acts trivially on the upper half-plane). Let $\mathcal{S}$ denote the strip $\left\{|\Re(\tau)| \leq \frac{1}{2}\right\}$ and let $\mathcal{F} = \left\{\tau \in \mathbb{H} \mid |\tau| \geq 1, |\Re(\tau)| \leq \frac{1}{2}\right\}$.
   (a) Show that $\mathcal{S}$ is a fundamental domain for $\Gamma_\infty\backslash\mathbb{H}$, hence surjects on $\Gamma\backslash\mathbb{H}$.
   (b) Let $\tau = x + iy \in \mathcal{S}$. Show that there are only finitely many $y' \geq y$ such that there exists x' for which $\tau' = x' + iy' \in \mathrm{SL}_2(\mathbb{Z}) \cdot \tau$.
   Hint: Recall that $y(\gamma\tau) = \frac{y(\tau)}{|c\tau+d|^2}$, and consider the real and imaginary parts of $c\tau + d$ separately.
   (c) Let $f\colon \mathcal{S} \to \mathcal{S}$ be as follows: if $|\tau| \geq 1$ set $f(\tau) = \tau$. Otherwise, let $f(\tau) = T^m S\tau$ with m chosen so that $f(\tau) \in \mathcal{S}$. Show that $\Im(f(\tau)) > \Im(\tau)$.
   (d) Conclude that $\mathcal{F}$ surjects on $\Gamma\backslash\mathbb{H}$.
   (e) Let $\tau \in \mathcal{F}$ and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ be such that $\gamma\tau \in \mathcal{F}$ but $\gamma \neq \pm I$. Show that one of the following holds:
      (1) $|\Re(\tau)| = \frac{1}{2}$ and $\gamma \in \left\{\pm T, \pm T^{-1}\right\}$.
      (2) $|\tau| = 1$ and $\gamma \in \{\pm S\}$.
      (3) $\tau = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$
   (f) Show that $-I \in \Gamma$ and conclude that $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and that $\mathcal{F}$ is a fundamental domain for $Y(1) = \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$.

OPT Let $E = \mathbb{C}/\Lambda$ be an elliptic curve.
   (a) Show that up to isomorphism of elliptic curves we may assume that $1 \in \Lambda$ and that it is a non-zero element of minimal length.
   (b) Let $\tau \in \mathbb{H} \cap \Lambda$ be of minimal norm. Show that $|\tau| \geq 1$ and that $|\Re(\tau)| \leq \frac{1}{2}$, that is that $\tau \in \mathcal{F}$.
   (c) Show for any $z \in \mathbb{C}$ there is $z' \in z + \Lambda_\tau$ with $|z'| < \frac{1}{2} + \frac{1}{2}|\tau| \leq |\tau|$ and conclude that $\Lambda = \Lambda_\tau$, that is that $\mathcal{F}$ surjects on $Y(1)$.
   – Using 1(e) it follows again that $\mathcal{F}$ is a fundamental domain.

3. Let $dA(\tau) = \frac{dxdy}{y^2}$ denote the hyperbolic area measure on $\mathbb{H}$. Calculate $\int_{\mathcal{F}} dA(\tau)$.

The moduli space of elliptic curves with level structure

4. Let $\Lambda < \mathbb{C}$ be a lattice, $E = \mathbb{C}/\Lambda$ the associated elliptic curve. For an integer N write $E[N]$ for the N-*torsion points*, that is the points $x \in E$ such that $N \cdot x = 0$.
   (a) Show that $E[N] \simeq (\mathbb{Z}/N\mathbb{Z})^2$ as abelian groups.
   – We now study the action of $G = \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ on $E[N]$.
   (b) Show that G acts transitively on the set of points in $E[N]$ whose order is N exactly. Find the stabilizer of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (call it $K_1(N)$) and the number of such points.
   (c) Conclude that G acts transitively on the set of subgroups of $E[N]$ which is cyclic of order N. Find the stabilizer of the subgroup $\left\{ \begin{pmatrix} * \\ 0 \end{pmatrix} \right\}$ (call it $K_0(N)$) and the number of such subgroups.
   (d) Find the order of $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$. Write in the form $N^3 \prod_{p|N} f(p)$.

5. Let $Y_0(N)$ denote the set of isomorphism classes of pairs (E, C) where E is a complex elliptic curve and $C \subset E$ is a subgroup isomorphic to C_N ($(E,C) \sim (E,C')$ if there exists an isomorphism $f \colon E \to E'$ such that $f(C) = C'$).
   (a) Show that the map $\mathbb{H} \to Y_1(N)$ mapping τ to the class of the pair $\left(\mathbb{C}/\Lambda_\tau, \frac{1}{N}\mathbb{Z}/\mathbb{Z}\right)$ (i.e. the subgroup of $\mathbb{C}/\Lambda_\tau$ generated by $\frac{1}{N} + \Lambda_\tau$) is surjective.
   (b) By analyzing the isomorphism relation show that $Y_0(N) = \Gamma_0(N)\backslash\mathbb{H}$ where $\Gamma_0(N)$ is the inverse image in $\mathrm{SL}_2(\mathbb{Z})$ of $K_0(N)$.

OPT Let $Y_1(N)$ denote the set of isomorphism classes of pairs (E, P) where E is a complex elliptic curve and $P \in E[N]$ has order N exactly.
   (a) Show that the map $\mathbb{H} \to Y_1(N)$ mapping τ to the class of the pair $\left(\mathbb{C}/\Lambda_\tau, \frac{1}{N} + \Lambda_\tau\right)$ is surjective.
   (b) By analyzing the isomorphism relation show that $Y_1(N) = \Gamma_1(N)\backslash\mathbb{H}$ where $\Gamma_1(N)$ is the inverse image in $\mathrm{SL}_2(\mathbb{Z})$ of $K_0(N)$.

OPT Let $Y(N)$ denote the set of isomorphism classes of triples (E, P, Q) where E is a complex elliptic curve and $P, Q \in E[N]$ are an ordered basis for $E[N]$ as a free $\mathbb{Z}/N\mathbb{Z}$-module.
   (a) Show that the map $\mathbb{H} \to Y(N)$ mapping τ to the class of the triple $\left(\mathbb{C}/\Lambda_\tau, \frac{1}{N}\mathbb{Z} + \Lambda_\tau, \frac{\tau}{N} + \Lambda_\tau\right)$ is surjective.
   (b) By analyzing the isomorphism relation show that $Y(N) = \Gamma(N)\backslash\mathbb{H}$ where $\Gamma(N)$ is the kernel of the map $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$.

Hyperbolic Convergence Lemma

Let $\Gamma < \mathrm{SL}_2(\mathbb{R})$ be discrete and assume that $\Gamma_\infty = \Gamma \cap P$ is non-trivial (i.e. infinite), with the image in $\mathrm{PSL}_2(\mathbb{R})$ generated by $\begin{pmatrix} 1 & h \\ & 1 \end{pmatrix}$.

8. (Counting Lemma)
   (a) Show that a fundamental domain for $\Gamma_\infty \backslash \mathbb{H}$ is the strip $\left\{ |\Re(z)| \leq \frac{h}{2} \right\}$.
   (b) Calculate the hyperbolic area of the half-strip $\left\{ x + iy \mid |x| \leq \frac{h}{2}, y \geq \frac{1}{Y} \right\}$.
   (c) For $z \in \mathbb{H}$ show that there exists $C > 0$ (depending locally uniformly on z) such that for all $Y > 0$, $\#R_Y \leq C(1+Y)$ where
   $$R_Y = \left\{ \Gamma_\infty \gamma \in \Gamma_\infty \backslash \Gamma \mid y(\gamma z) \geq \frac{1}{Y} \right\}.$$
   Hint: Let B be a hyperbolic ball around z of small enough radius so that if $\gamma \in \Gamma$ satisfies $\gamma B \cap B \neq \emptyset$ then γ belongs to the finite group Γ_z, and consider the set of images of $\Gamma \cdot B$ in the strip.

For $\Re(s) > 1$ we define the *non-holomorphic Eisenstein series* to be
$$E(z;s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} y(\gamma z)^{-s}$$

9. (Convergence Lemma)
   (a) Show that the series $E(z;\sigma)$ converges absolutely if $\sigma > 1$.
   Hint: Show that $E(\sigma;z) \leq A + \sum_{n=1}^{\infty} (\#R_{n+1} - \#R_n)\, n^{-\sigma}$ where A is easily controlled. Now use summation by parts.
   (b) Conclude that $E(z;s)$ extends to a holomorphic function of s in $\Re(s) > 1$.

2.2. Discrete subgroups and fundamental domains

DEFINITION 51. G acts *properly discontinuously* if for all $z \in \mathbb{H}$, K cpt, $\{\gamma \in \Gamma \mid \gamma z \in K\}$ is finite.

LEMMA 52. *G acts properly discontinuously iff for all z there is an open nbd V such that $\#\{\gamma \mid \gamma V \cap V \neq \emptyset\} < \infty$ iff for all A, B cpt $\#\{\gamma \mid \gamma A \cap B \neq \emptyset\} < \infty$.*

PROOF. If $\gamma B(z,\varepsilon) \cap B(z,\varepsilon) \neq \emptyset$ then $d(z,\gamma z) \leq 2\varepsilon$. Now cover A with finitely many V_z. □

COROLLARY 53. *For all z there is a nbd W such that $\gamma W \cap W \neq \emptyset \Rightarrow \gamma z = z$.*

PROOF. Take V as in Lemma. Now if $d(\gamma z, z) = \varepsilon$ then $\gamma B(z,\varepsilon/2) \cap B(z,\varepsilon/2) = \emptyset$. □

THEOREM 54. *$\Gamma \subset G$ acts properly discontinuously iff $\Gamma < G$ is discrete.*

PROOF. If $\gamma_n \to \mathrm{id}$ then $\gamma_n z \to z$ and then by the Corollary $\gamma_n \in \Gamma_z$ for n large, and Γ_z is finite so γ_n is eventually constant. If $\Gamma < G$ is discrete and $\gamma_n z \in K$ then can have $\gamma_n z \to w$. Wlog $z = i$ and writing $\gamma_n = p_n k_n$ we have $p_n \to p$ such that $pi = w$ and we can assume $k_n \to k$ since K is compact. It follows that $\gamma_n \to pk$ which contradicts discreteness unless sequence is eventually constant. □

THEOREM 55. *$z \in \mathbb{H}$ with Γ_z trivial. Then $\mathcal{F}_D$ is a fundamental domain. It is convex, and its boundary is a union of geodesic segments.*

PROOF. $\mathcal{F}_z$ is the intersection of countably many closed half-planes hence closed and convex. $\mathcal{F}_z \cap B(z,R)$ is determined by finitely many half-planes, so $\partial \mathcal{F}_z$ is a countable union of geodesic segments, hence of zero area, and $\mathcal{F}_z$ is the closure of its interior (take radial geodesics to analyze the boundary). If $Tw, w \in \mathcal{F}_{\mathrm{D}}$ for $T \neq \mathrm{Id}$ then $d(w,z) = d(w, T^{-1}z)$ so w is on the boundary of $\mathcal{F}_{\mathrm{D}}$. □

LEMMA 56. *Every Γ-orbit intersects $\mathcal{F}_D$ finitely many times.*

PROOF. Let $w \in \mathcal{F}_{\mathrm{D}}$. If $\gamma w \in \mathcal{F}_{\mathrm{D}}$ then $d(\gamma w, z) = d(w,z)$ and can have at most finitely many such. □

COROLLARY 57. *$\mathcal{F}_D$ is* locally finite*: on finitely many translates intersect a compact set K. Hence:*

(1) *Every elliptic orbit has a representative in $\partial \mathcal{F}_D$. If the period is k the angle is at most $\frac{2\pi}{k}$.*
(2) *Γ is generated by elements that match sides. If $\mathcal{F}_D$ is geom finite Γ is f.g.*

PROOF. (1) Folllows from Lemma. (2) Consider the graph with vertices $\gamma \mathcal{F}_{\mathrm{D}}$ and edges from these elements. It is connected, since if two share a vertex then at the vertex there are finitely many translates, all sharing sides. □

DEFINITION 58. A *vertex* of $\partial \mathcal{F}_{\mathrm{D}}$ is the intersection of two boundary geodesics, or the fixed point of an involution in the middle of a geodesic.

LEMMA 59. *The set of vertices is discrete*

PROOF. Say w is a vertex on the side $L(z,\gamma z)$. Then $d(z,\gamma z) \leq 2d(z,w)$ so at most finitely many γ. □

PROPOSITION 60. *Let $\{v_i\}$ be a Γ-orbit of vertices, and let Γ_{v_1} have order m (perhaps $m = 1$). Let ω_i be the interior angles at v_i. Then $\sum_i \omega_i = \frac{2\pi}{m}$.*

PROOF. Let γ_i be such that $\gamma_i v_1 = v_i$ and let σ generate Γ_{v_1} then $\left\{\gamma_i \sigma^j \mathcal{F}_{\mathrm{D}}\right\}$ is the set of fundamental domains containing v_1. The total angle they have at v_1 is $2\pi = m\sum_i \omega_i$. $\square$

THEOREM 61. *(Siegel) Let $\mu(\mathcal{F}_D) < \infty$. Then $\mathcal{F}_D$ is geometrically finite.*

PROOF. Let a chain of finite vertices by w_k, with $[w_k, w_{k+1}]$ being a side. Form triangles with z and let the angles be $\alpha_k, \beta_k, \gamma_k$ so $\omega_k = \gamma_{k-1} + \beta_k$. Then the area of kth triangle is $\pi - (\alpha_k + \beta_k + \gamma_k)$. Adding for $-N+1 \leq k \leq M$ find $\sum_{k=-N+1}^{M} (\pi - \omega_k) + \pi - \beta_{-N} - \gamma_M \leq \sum_{k=-N}^{k=M} \alpha_k + \sum_{k=-N}^{M} \mu(\Delta_k)$. RHS is bounded above so sum on LHS converges. Also, infinitely often w_{k+1} is farther away than w_k so $\beta_k \geq \gamma_k$ and $\gamma_k \leq \frac{\pi}{2}$. Similarly on the other side. Summing over all chains get

$$\sum_v (\pi - \omega_v) \leq 2\pi + \mu(\mathcal{F}_{\mathrm{D}}).$$

Now separate the sum over orbits of vertices. An orbit of period $m = 1$ and length n has angles $\omega_{v_i} < \pi$ and $\sum \omega_{v_i} = 2\pi$ so $\sum(\pi - \omega_{v_i}) = (n-2)\pi \geq \pi$. An orbit of period $m \geq 3$ has $\sum(\pi - \omega_i) = n\pi - \frac{2\pi}{m} = \left(n - \frac{2}{m}\right)\pi \geq \frac{\pi}{3}$. Any elliptic fixed point of order 2 is the middle of a segment with endpoints of other types. It follows that there are finitely many vertices in $\mathbb{H}$. For the ideal vertices take the polygon with just those vertices. It has area πN where N is their number, so there are only finitely many of those too. $\square$

THEOREM 62. *Let $\mathcal{F}_z$ be geometrically finite. Then Γ is finitely generated.*

PROOF. Let $S = \{s \in \Gamma \mid s\mathcal{F}_{\mathrm{D}} \cap \mathcal{F}_{\mathrm{D}} \neq \emptyset\}$. $\square$

2.3. Cusps

PROPOSITION 63. *If $\mathcal{F}_D$ is compact there are no parabolic elements.*

PROOF. Let $\eta(z) = \inf\{d(z, \gamma z) \mid \gamma \text{ not elliptic}\}$. This is γ-invariant and continuous, and positive pointwise. It is therefore uniformly bounded below. $\square$

PROPOSITION 64. *If $\mathcal{F}_D$ is not compact it contains a vertex at infinity and $\Gamma\backslash\mathbb{H}$ is not compact.*

PROOF. Let $r(\theta)$ be the radius of $\mathcal{F}_{\mathrm{D}}$ in the direction θ. This is continuous if finite, so if $\mathcal{F}_{\mathrm{D}}$ is non-compact $\mathcal{F}_{\mathrm{D}}$ touches the boundary and contains an embedded infinite geodesic ray, hence isn't compact. $\square$

DEFINITION 65. Consider the action of a discrete subgroup $\Gamma < \mathrm{SL}_2(\mathbb{R})$ on $\partial\mathbb{H} = \mathbb{P}^1(\mathbb{R})$. We say that $\xi \in \mathbb{P}^1(\mathbb{R})$ is a *cusp* Γ *if* $\Gamma_\xi = \mathrm{Stab}_\Gamma(\xi)$ is non-trivial (contains an element other than $\pm I$). We say two cusps are *equivalent* if $\gamma\xi = \eta$ for some $\gamma \in \Gamma$.

THEOREM 66. *Let $\mathcal{F}_D$ be non-compact but of finite volume. Then there are vertices at infinity, they are parabolic fixed points, and cusps of Γ are equivalent to vertices at infinity.*

PROOF. The endpoint of the geodesic ray in Proposition 64 cannot line on a side at infinity so must be a vertex. Let ξ be any vertex at infinity. Consider $\{\gamma \in \Gamma \mid \xi \in \partial(\gamma\mathcal{F}_{\mathrm{D}})\}$, certainly infinite. Then $\gamma^{-1}\xi$ is also a vertex, hence lies in a finite set. It follows that that Γ_ξ is infinite. If $\gamma \in \Gamma_\xi$ is not trivial then it must be parabolic: if $z, \gamma z$ are on the same horosphere γ is parabolic. Otherwise can assume γz is inside the horoball. Then $h(t) = [\gamma z, \xi]_t$ is always inside $g(t) = [z, \xi]_t$ so that $l = \lim_{t\to\infty} d(g(t), h(t)) > 0$. Then $d(g(t), h(t-l)) \to 0$ which means $d(g(t), \gamma z) \approx d(g(t), z) - l$, which isn't possible.

Conversely, assume that ∞ is a cusp of width h. We show that a cuspidal neighbourhood embeds in the quotient. Assume that $\Im(\gamma z) \geq \Im(z) \geq Y$ and both have real parts at most $\frac{h}{2}$. From

$$\frac{y}{|cz+d|^2} \geq y$$

and hence $|cz+d|^2 \leq 1$, we get $c \leq \frac{1}{Y}$. Now consider the sequence $\gamma_0 = \gamma$, $\gamma_{n+1} = \gamma_n \begin{pmatrix} 1 & h \\ & 1 \end{pmatrix} \gamma_n^{-1}$. If $|ch| \leq 1$ this converges to the identity unless $c = 0$.

It now follows that if ξ, η are inequivalent cusps then deep enough cuspidal neighbourhood are disjoint. Now let ξ be a cusp inequivalent to the vertices. Then its cuspidal neighbourhood is disjoint from those of the vertices. But $\mathcal{F}_{\mathrm{D}}$ minus thoses is compact. □

2.4. Construction of $\Gamma\backslash\mathbb{H}$

The orbifold structure is automatic, together with the Riemannian metric. For a Riemann surface structure, identify $\mathbb{H}$ with the disc model so that a given $z \in \mathbb{H}$ maps to 0. Then Γ_z is rotation by an angle, and take the co-ordinate z^k where k is the period. Near a cusp take the co-ordinate $q_h = e^{2\pi i a/h}$.

DEFINITION 67. The *periods* of $\Gamma\backslash\mathbb{H}$ are the orders of the elliptic points and the number of cusps.

EXAMPLE 68. The function field of $Y(1)$: a contour integral argument shows that $j\colon X(1) \to \mathbb{P}^1(\mathbb{C})$ is bijective. Now let $f \in \mathbb{C}(X(1))$ and let $g = \frac{\prod_k (j(z)-j(p_k))}{\prod_l (j(z)-j(q_l))}$ have the same zeroes and poles as f in $Y(1)$. Then $\frac{g}{f}$ has no zeroes or poles in $Y(1)$. But since $X(1)$ is compact, $\frac{g}{f}$ has neither a zero nor a pole at ∞ so $\frac{g}{f}$ is constant.

CHAPTER 3

Modular forms

Problems:

3.1. Holomorphic forms

The action of $\mathrm{GL}_2^+(\mathbb{R})$ on $\mathbb{H}$extends to an action on functions: set

$$(f|_k\gamma)(z) = (\det\gamma)^{k/2}(cz+d)^{-k}f(\gamma z)\,.$$

PROBLEM 69. Verify that this is a right action.

Fix a lattice $\Gamma < \mathrm{SL}_2(\mathbb{R})$.

DEFINITION 70. For an integer k set $\Omega_k(\Gamma) = \{f\colon \mathbb{H} \to \mathbb{C} \mid f \text{ meromorphic}, \forall\gamma\in\Gamma : f|_k\gamma = f\}$. Call these *automorphic forms*.

PROBLEM 71. Let $f \in \Omega_k(\Gamma)$, $g \in \Omega_l(\Gamma)$. Show that $fg \in \Omega_{k+l}(\Gamma)$ and that if f is non-zero, $f^{-1} \in \Omega_{-k}(\Gamma)$. Show that the sum in $\Omega(\Gamma) = \oplus_k\Omega_k(\Gamma)$ is direct, so that Ω is a graded field, containing the field Ω_0.

Now let $\xi \in \partial\mathbb{H}$ be a cusp of Γ, and let $\sigma \in \mathrm{SL}_2(\mathbb{R})$ such that $\xi = \sigma\infty$. Say that $\sigma^{-1}\Gamma_\xi\sigma \cdot \{\pm 1\} = \left\{\pm\begin{pmatrix} 1 & h \\ & 1\end{pmatrix}^n\right\}$. Let $f \in \Omega_k(\Gamma)$ with k even. Then $f|_k\sigma$ is a meromorphic function invariant by $\begin{pmatrix} 1 & h \\ & 1\end{pmatrix}$, and hence a meromorphic function on the punctured disc $\left\langle\begin{pmatrix} 1 & h \\ & 1\end{pmatrix}\right\rangle\backslash\mathbb{H}$, that is $f|_k\sigma = g(q_h)$ for some g meromorphic in $\mathbb{D}$ where $q_h = e(z/h)$.

DEFINITION 72. Say that f is *meromorphic*, *holomorphic* or *vanishes* at ξ if g is so at $q = 0$, and set:

(1) $\mathcal{A}_k(\Gamma) = \{f \in \Omega_k(\Gamma) \mid f \text{ is meromorphic at the cusps}\}$.
(2) $\mathcal{H}_k(\Gamma) = \{f \in \mathcal{A}_k(\Gamma) \mid f \text{ is holomorphic on } \mathbb{H}\text{and at the cusps}\}$
(3) $\mathcal{S}_k(\Gamma) = \{f \in \mathcal{H}_k(\Gamma) \mid f \text{ vanishes at the cusps}\}$.

For $f \in \mathcal{A}_k(\Gamma)$ g will have a Laurent expansion near $q = 0$, so we have a *Fourier expasion* $(f|_k\sigma)(z) = \sum_{n\in\mathbb{Z}} a_n q_h^n$ where $q_h(z) = e(z/h)$.

PROBLEM 73. This expansion is independent of the choice of σ.

REMARK 74. For k odd one says f is meromorphic, holomorphic or cuspidal if f^2 is so. We will not discuss modular forms of odd weight.

PROBLEM 75. $\oplus_k\mathcal{A}_k(\Gamma)$ is a graded field. $\mathcal{A}_0(\Gamma)$ is the function field of the Riemann surface $\Gamma\backslash\mathbb{H}^*$.

DEFINITION 76. $f \in \Omega_k(\Gamma)$ is a *weak modular form* if it is holomorphic in $\mathbb{H}$.

LEMMA 77. *Let $f \in \Omega_k(\Gamma)$ be a weak modular form. Assume that $f(z) = O(y(z)^{-\nu})$ for some $\nu \in \mathbb{R}_{\geq 0}$. Then $f \in HkG$. If $\nu < k$ then $f \in \mathcal{S}_k(\Gamma)$.*

PROOF. Let ξ be a cusp of Γ, $\xi \neq \infty$. Then $\sigma_\xi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $c \neq 0$. Then $\Im(\sigma_\xi z) = \frac{\Im(z)}{|cz+d|^2} = \frac{\Im(z)}{c^2\Im(z)^2 + (c\Re(z)+d)^2} \gg \frac{1}{\Im(z)}$ uniformly in $\Re(z)$ and $\frac{1}{(cz+d)^k} \ll \Im(z)^{-k}$. It follows that

$$\begin{aligned} a_n &= \frac{1}{h}\int_{iy_0}^{iy_0+h} e(-nz/h)\left(f|_k\sigma_\xi\right)(z)dz \\ &\ll e^{2\pi n y_0/h} y_0^{\nu-k}. \end{aligned}$$

Now if $n < 0$ taking $\Im(z) \to \infty$ shows $a_n = 0$ and if $\nu < k$ this will also work for $n = 0$. Since ∞ is equivalent to a real cusp we are done. □

THEOREM 78. *$f \in \Omega_k(\Gamma)$ is a cusp form iff $f(z)y^{k/2}$ is bounded on $\mathbb{H}$.*

PROOF. Sufficiency is by the Lemma. For necessity note first that if $g = f|_k\alpha$ then $\left|g(z)y^{k/2}\right| = \left(\frac{y}{|cz+d|^2}\right)^{k/2}|f(\alpha z)| = |f(\alpha z)|\, y(\alpha z)^{k/2}$ so the condition is invariant under conjugation. Now let $i\infty$ be a cusp of Γ with expansion $f(z) = \sum_{n\geq 1} a_n q_h^n$. Then from absolute convergence, $|f(z)| = O(q_h)$ as $\Im(z) \to \infty$ which decays exponentially and we are done. □

COROLLARY 79. *(Hecke's "Trivial" bound) let $f \in \mathcal{S}_k(\Gamma)$. Then at every cusp the Fourier expansion satisfies $a_n = O(n^{k/2})$.*

PROOF. Wlog may assume the cusp is at infinity. We then have

$$\begin{aligned} a_n &= \frac{1}{h}\int_{iy_0}^{iy_0+h} f(z)e(-nz/h)dz \\ &\ll e^{2\pi n y_0/h} y_0^{-k/2}. \end{aligned}$$

Now take $y_0 = \frac{1}{n}$. □

REMARK 80. Quantitative equidistribution of closed horocycles gives a better bound than Cauchy-Schwartz and hence a saving in the exponent.

More generally, if $\chi\colon \Gamma \to S^1$ is of finite order then set $\Omega_k(\Gamma,\chi) = \{f \in \Omega_k(\operatorname{Ker}\chi) \mid \forall\gamma\in\Gamma : f|_k\gamma = \chi(\gamma)f\}$ and similarly for the subspaces. If $f,g \in \mathcal{H}_k(\Gamma,\chi)$ with at least one a cusp form then $fg \in \mathcal{S}_{2k}(\Gamma,\chi^2)$ and $|fg|\,y^k$ is bounded and Γ-invariant.

DEFINITION 81. The *Patterson inner product* is

$$(f,g) = \frac{1}{\operatorname{vol}(\Gamma\backslash\mathbb{H})}\int_{\Gamma\backslash\mathbb{H}} f(z)\overline{g(z)}y^k dA(z).$$

This makes $\mathcal{S}_k(\Gamma,\chi)$ into an inner product space. Note that this definition is independent of the choice of Γ.

DEFINITION 82. $\mathcal{E}_k(\Gamma,\chi)$ is the orthogonal complement of $\mathcal{S}_k(\Gamma,\chi)$ in $HkGx$.

FACT 83. *(Riemann-Roch) Let g be the genus of* Γ, $\{e_u\}$ *the orders of the elliptic points and t the number of cusps. Then for k even,*

$$\dim \mathcal{S}_k(\Gamma) = \begin{cases} (k-1)(g-1) + \frac{k}{2}\sum_u \left(1 - \frac{1}{e_u}\right) + \left(\frac{k}{2} - 1\right)t & k > 2 \\ g & k = 2 \\ 1 & k = 0, t = 0 \\ 0 & k = 0, t > 0 \\ 0 & k < 0 \end{cases}$$

and

$$\dim \mathcal{H}_k(\Gamma) = \begin{cases} \dim \mathcal{S}_k(\Gamma) + t & k > 2 \\ \dim \mathcal{S}_k(\Gamma) + t - 1 & k = 2, t > 0 \\ \dim \mathcal{S}_k(\Gamma) & k = 2, t = 0 \\ 1 & k = 0 \\ 0 & k < 0 \end{cases}.$$

EXAMPLE 84. $Y(1)$ has one elliptic point of order 2, one elliptic point of order 3 and one cusp.

Math 613: Problem set 4 (due 18/10/09)

The weight-k action

OPT For a field F let $\mathbb{P}^1(F)$ denote the set of 1-dimensional subspaces of F^2. Write $\begin{bmatrix} a \\ b \end{bmatrix}$ for the subspace generated by the vector $\begin{pmatrix} a \\ b \end{pmatrix} \in F^2 \setminus \{0\}$.

(a) Show that a set of representatives for $\mathbb{P}^1(F)$ is given by $\left\{ \begin{bmatrix} z \\ 1 \end{bmatrix} \right\}_{z \in F}$ togther with the "point at infinity" $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ which we can also denote $\begin{bmatrix} \infty \\ 1 \end{bmatrix}$.

(b) Show that the action of $\mathrm{GL}_2(F)$ on F^2 induces an action on $\mathbb{P}^1(F)$, given in co-ordinates by $g \begin{bmatrix} z \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{az+b}{cz+d} \\ 1 \end{bmatrix}$ (don't forget the case $z = \infty$).

— Let $j(g,z) = cz + d$ so that $g \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} g \cdot z \\ 1 \end{pmatrix} j(g,z)$. For f defined on $\mathbb{P}^1(F)$ set (formally) $(f|_k g)(z) = f(g \cdot z) j(g,z)^{-k}$.

(c) Show that $j(g \cdot g', z) = j(g, g' \cdot z) j(g', z)$.

(d) Show that $f \mapsto f|_k g$ is a right action of $\mathrm{SL}_2(F)$.

(e) Using $\det \begin{pmatrix} z + dz & z \\ 1 & 1 \end{pmatrix} = dz$ show that $d(gz) = \frac{1}{j(g,z)^2} dz$ as formal differentials on $\mathbb{P}^1(F)$.

2. (Linear independence) We now specialize to the case of $\mathrm{SL}_2(\mathbb{R}) \to \mathrm{PGL}_2(\mathbb{C})$ acting on $\mathbb{P}^1(\mathbb{C})$, where the action restricts to an action on $\mathbb{H}$.
   (a) Show that $j(g,z) \neq 0, \infty$ for $g \in \mathrm{SL}_2(\mathbb{R})$, $z \in \mathbb{H}$, so that the formal calculation of part 1 applies here.
   (b) Show that $j(g,z) = j(g',z)$ as functions iff $g'g^{-1} \in P$.
   (c) Let $\Gamma < \mathrm{SL}_2(\mathbb{R})$ be a discrete subgroup and assume that $\Gamma_\infty = \Gamma \cap P$ is of infinite index in Γ. Find $\{\gamma_m\}_{m=1}^{\infty} \subset \Gamma$ such that $\{j(\gamma_m, z)\}$ are distinct functions.
   (d) Choose $f_k \in \Omega_k(\Gamma)$ for each k (such that all but finitely many are zero) and assume that $\sum_k f_k = 0$. Show that for each m we have $\sum_k j(\gamma_m, z)^k f_k(z) = 0$.
   (e) Show that for m large enough the system of linear equations above for f_k is invertible, and conclude that each $f_k = 0$.
   (f) Conclude that the sum $\sum_k \Omega_k(\Gamma)(\Gamma)$ is direct.

More on cusps

3. Let Γ be a Fuchsian group of the first kind, $X_\Gamma = \Gamma \backslash \mathbb{H}^*$ its associated closed Riemann surfacethe element $T = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{R})$.
   (a) Let $\xi \in \partial\mathbb{H}$ be a cusp of Γ, and let $\gamma \in \Gamma \setminus \Gamma_\xi$. Show that the set of z for which the geodesic $L(z, \gamma z) = \{w \mid d(w,z) = d(w, \gamma z)\}$ ends at ξ is a Euclidean circle. Conclude that for almost all z, for all $\gamma \neq$ idwe have $\gamma z \neq z$ and if line $L(z, \gamma z)$ touches a cusp ξ then $\gamma \in \Gamma_\xi$.

– Fix such z and let $\mathcal{F}_{\mathrm{D}}$be its associated Dirichlet domain.

(b) Let $\xi,\eta\in\partial\mathbb{H}$ be two vertices at infinity of $\mathcal{F}_{\mathrm{D}}$. Show that they are Γ-inequivalent.

— Conclude that the vertices at infinity $\{\xi_k\}_{k=1}^{K}$ of $\mathcal{F}_{\mathrm{D}}$ are representatives for the $\Gamma-$equivalence classes of cusps of Γ.

(c) For each k let $\sigma_k\in\mathrm{SL}_2(\mathbb{R})$ be such that $\sigma_k\infty=\xi_k$. Show that $\sigma_k^{-1}\mathcal{F}_{\mathrm{D}}\cap\{y(z)>Y\}$ is a vertical strip $[x_0,x_0+h]\times(Y,\infty)$ for Y large enough.
Hint: Consider the two sides meeting at the vertex ξ_k.

OPT Show that we can choose σ_k such that $\sigma_k^{-1}\mathcal{F}_{\mathrm{D}}\cap\{y(z)>Y\}=\left[-\frac{1}{2},\frac{1}{2}\right]\times(Y,\infty)$ and that in that case the image of $\sigma_k^{-1}\Gamma_{\xi_k}\sigma_k$ in $\mathrm{PSL}_2(\mathbb{R})$ is the group generated by T.

(e) Set $\mathcal{F}_{k,Y}=\sigma_k\left[-\frac{1}{2},\frac{1}{2}\right]\times(Y,\infty)$ and $\mathcal{F}_Y=\mathcal{F}_{\mathrm{D}}\setminus\bigcup_k\mathcal{F}_{k,Y}$. Show that for Y large the $\mathcal{F}_{k,Y}$ are disjoint and $\mathcal{F}_Y$ is compact.

4. The *invariant height* on $\Gamma\backslash\mathbb{H}$ is defined by

$$y_\Gamma(z)=\max_k\max_{\gamma\in\Gamma}y\left(\sigma_k\gamma z\right).$$

(a) Show that $\max_{\gamma\in\Gamma}y\left(\sigma_k\gamma z\right)$ is finite and continuous.
Hint: By problem set 3, problem 8(c) the set of y-values is discrete and bounded above.

(b) Show that y_Γ is a continous Γ-invariant function on $\mathbb{H}$. Show that $y_\Gamma(z_n)\to\infty$ if z_n approach a cusp.

(c) Show that $\{z\in\Gamma\backslash\mathbb{H}\mid y_\Gamma(z)\le Y\}$ is compact, and that if $y_\Gamma(z_n)\to\infty$ then there is a subsequence which converges to a cusp.
Hint: The first part is variant of 3(d).

5. Let $f\in\mathcal{A}_0(\Gamma)=\mathbb{C}(X_\Gamma)$ be a meromorphic function on X_Γ.

(a) Show that for Y large enough f has no zeroes or poles in the region $y_\Gamma(z)>Y$.

— Assume now that Y is also large enough for 3(d) to hold. Let C_Y be the contour that goes along the boundary of $\mathcal{F}_{\mathrm{D}}$ except that at each cusps one truncates the cusp along the curve $y_\Gamma=Y$, and write $C_Y=C_0\bigcup\cup_kC_k$ where $C_0=C_Y\cap\partial\mathcal{F}_{\mathrm{D}}$ and C_k is the closed horocycle at the kth cusp.

(b) Show that $\frac{1}{2\pi i}\oint_{C_0}\frac{f'}{f}dz=0$ using the side-pairings and the invariance of y_Γ.

(c) Evaluate $\frac{1}{2\pi i}\int_{C_k}\frac{f'}{f}dz$ in terms of the behaviour of f at ξ_k by mapping the cusp neighbourhood to a punctured disk.

(d) Since $\frac{1}{2\pi i}\oint_{C_Y}\frac{f'}{f}dz$ counts the zeroes and poles in $\mathcal{F}_Y$, show that f has the same number of zeroes and poles in X_Γ.

6. $X(1)=\Gamma(1)\backslash\mathbb{H}^*$. We have seen in class that $j\colon X(1)\to\mathbb{P}^1(\mathbb{C})$ is a biholomorphism. In particular, all values are simple.

(a) Let $f\in\mathcal{A}_0(\Gamma(1))$ be non-constant. Construct $g\in\mathbb{C}(j)$ such that f,g have the same zeroes and poles in $Y(1)$.
Hint: $j(z)-j(z_0)$ has a simple zero at z_0, a pole at the cusp, and no other zeroes or poles.

(b) Show that $\frac{f}{g}$ has no zeroes or poles in $Y(1)$, and conclude that it has no zeroes or poles in $X(1)$.

(c) Applying the maximum principle show that $\frac{f}{g}$ is constant and conclude that $\mathbb{C}(X(1))=\mathcal{A}_0(\Gamma(1))=\mathbb{C}(j)$.

On the choice of σ_ξ

7. Let Γ be a Fuchsian group with a cusp ξ, and let $\sigma, \sigma' \in \mathrm{SL}_2(\mathbb{R})$ such that $\sigma\infty = \sigma'\infty = \xi$. Let $f \in \Omega_k(\Gamma)$.
   (a) Show in the definition of f being meromorphic/holomorphic/vanishing at ξ using σ or σ' would not change the conclusion.
   (b) Assume that f is meromorphic at ξ or holomorphic on $\mathbb{H}$. In either case show that the Fourier expansion of f at ξ is essentially independent of the choice σ or σ'. Is the expansion truly independent of the choice?

The cusps of congruence subgroups

8. Let Γ be a Fuchsian group, and let Γ' be a subgroup of finite index.
   (a) Show that Γ and Γ' have the same cusps.
   (b) Let ξ be a cusp of Γ. Show that the Γ'-equivalence classes of cusps which are Γ-equivalent of ξ are in bijection with the double coset space $\Gamma' \backslash \Gamma / \Gamma_\xi$.
   (c) Let $\Gamma_N < \Gamma'$ be normal in Γ, and write bars for the image in the quotient group $\bar{\Gamma} = \Gamma_N \backslash \Gamma$. Show that the map $\Gamma \to \bar{\Gamma}$ induces a bijection $\Gamma' \backslash \Gamma_N / \Gamma_\xi \to \overline{\Gamma'} \backslash \bar{\Gamma} / \overline{\Gamma_\xi}$.

9. Let $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$ and recall its subgroups $\Gamma(N) < \Gamma_0(N) < \Gamma_1(N)$ from Problem set 3.
   (a) Show that the cusps of $\Gamma(1)$ are precisely $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\} \subset \mathbb{R} \cup \{\infty\} = \partial\mathbb{H}$, and that $\Gamma(1)$ acts transitively there.
   — Let $\Gamma_\infty = \Gamma(1)_{i\infty}$ and let $\Gamma_\infty^+ = \langle T \rangle$ where T is the translation.
   (b) Let $\bar{\Gamma} = \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$. Show that the map $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (c,d)$ induces a bijection between $\overline{\Gamma}/\overline{\Gamma_\infty^+}$ and the set of elements of order N in $(\mathbb{Z}/N\mathbb{Z})^2$.
   Hint: This was already done in PS3.
   (b) Show that $X_0(N) = X_{\Gamma_0(N)}$ has $\sum_{d|N} \phi\left((d, N/d)\right)$ cusps. In particular, for p prime $X_0(p)$ has two cusps – in this case find representatives.
   OPT Count the cusps of $X(N) = X_{\Gamma(N)}$ and $X_1(N) = X_{\Gamma_1(N)}$.

3.2. Eisenstein series and Poincaré series

EXERCISE 85. Write the Eisenstein series for $Y(1)$ as sums over $\Gamma(1)_\infty\backslash\Gamma(1)$.

LEMMA 86. *Let f be holomorphic in a domain $\Omega\subset\mathbb{C}$. Let $p\geq 1$ and assume that $\overline{B(z_0,3r)}\subset\Omega$. Then $f(z)\ll_{r,k,p}\left\|f\cdot y^{k/2}\right\|_{L^p(B(z_0,3r))}$ uniformly for $z\in\overline{B(z_0,r)}$.*

PROOF. Cauchy formula. □

COROLLARY 87. *(Banach spaces of holomorphic functions) If $\left\{f_n y^{k/2}\right\}_{n=1}^{\infty}\subset L^p(\Omega)$ converges in the L^p norm then the convergece is also locally uniform and the limit is $f\cdot y^{k/2}$ with f holomorphic.*

LEMMA 88. *Let f_n converge uniformly on compact subsets of $\Omega\setminus\{z_0\}$. Then also on Ω.*

PROOF. Cauchy integral formula. □

DEFINITION 89. Γ Fuchsian, χ character of finite order, $\Lambda<\Gamma$ subgroup. $\phi(z)\in\Omega_k(\Lambda,\chi)$ and has finitely many poles in $\Lambda\backslash\mathbb{H}$. Assume that $\phi(z)y^{k/2}$ has finite L^1-norm away from its poles on $\Lambda\backslash\mathbb{H}'$ where one takes away a cusp neighbourhood for each cusp of Γ. Set

$$F=\sum_{\gamma\in\Lambda\backslash\Gamma}\overline{\chi(\gamma)}\phi|_k\gamma.$$

PROPOSITION 90. *The series converges locally uniformly absolutely.*

PROOF. Take nbd W of z whose translates by Γ are disjoint and stay away from poles of ϕ and cusps. Then $\int_W\sum_\gamma|(\phi|_k\gamma)(z)|y^{k/2}dA(z)$ is bounded. Near a pole note that only finitely many translates are on a pole. □

PROPOSITION 91. *Assume: if x is not cusp of Λ, $\phi|_k\sigma\ll|z|^{-1-\varepsilon}$ ($\varepsilon>0$), near a cusp $\ll|z|^{-\delta}$($\delta\geq 0$) . Then f is holo at cusps, vanishes at cusps of Γ which are not cusps of Λ, and is a cusp form if $\delta>0$.*

PROOF. Fix a cusp x_0. For $\alpha\in\Lambda\backslash\Gamma/\Gamma_{x_0}$ set ϕ_α to be the sum for $\phi|_k\alpha$ over $\beta\in\alpha^{-1}\Lambda\alpha\cap\Gamma_{x_0}\backslash\Gamma_{x_0}$ (i.e. such that $\beta x_0=\alpha x_0$). Then ϕ_α is holo at x_0 and vanishes there if αx_0 is not a cusp of Λ or if $d>0$. F is defined by a sum which converges absolutely in compact neighbhourhoods of a cusp neighbhourhood punctured at the cusp, so it follows that F is holomorphic at cusps and vanishes at those which aren't cusps of Λ. If $\delta>0$ F is a cusp form. □

EXAMPLE 92. ∞ a cusp of Γ, $\phi_m(z)=e(\frac{m}{h}z)$, get $g_k^{(m)}(z)=P_m(z)$.

THEOREM 93. *Let $k\geq 3$. Then $g_k^{(m)}\in\mathcal{H}_k(\Gamma,\chi)$, vanishes at all cusps other than ∞, and also at ∞ if $m\geq 1$. For $m=0$ the constant coefficient at ∞ is 1.*

PROPOSITION 94. *Let $f_k\in S_k(\Gamma,\chi)$ have Fourier expansion $\sum_{n=1}^\infty a_n e(nz/h)$ at ∞. Then*

$$\left(g_k^{(m)},f\right)=\begin{cases}0 & m=0\\ a_m(4\pi m)^{1-k}h^k(k-2)! & m\geq 1\end{cases}$$

PROOF. Unfolding. □

THEOREM 95. $g_k^{(m)}$ *generate* $\mathcal{S}_k(\Gamma,\chi)$. *The Eisenstein series generate the complement.*

PROOF. By Proposition 94, a cusp form orthorgonal to all of $g_k^{(m)}$ vanishes. The Eisenstein series are clearly linearly independent; dimension count shows they have the right span. $\square$

3.3. Maass forms

Brief discussion; Fourier expansion; Eisenstein series; the spectral decomposition.

CHAPTER 4

Hecke Operators

4.1. Abstract Hecke algebras

4.1.1. Commensurators and Double cosets. Fix a group G.

DEFINITION 96. Call $\Gamma, \Gamma' < G$ *commensurable* if $\Gamma \cap \Gamma'$ has finite index in both and write $\Gamma \approx \Gamma'$. Say $g \in G$ *commensurates* Γ if $g\Gamma g^{-1} \approx \Gamma'$, and write $\mathrm{Comm}(\Gamma)$ for the set of elements that commensurate it.

EXAMPLE 97. If $\Gamma = \mathrm{SL}_2(\mathbb{Z}) \subset \mathrm{GL}_2(\mathbb{R}) = G$ then $\mathrm{Comm}(\Gamma) = \mathbb{R}^\times \cdot \mathrm{GL}_2(\mathbb{Q})$.

PROPOSITION 98. *(Commensurators)*

(1) *Commensurability is an equivalence relation, equivariant under conjugation in G.*
(2) $\mathrm{Comm}(\Gamma)$ *is a subgroup of* G *containing* Γ.
(3) *If* $\Gamma \approx \Gamma'$ *then they have the same commensurator.*

REMARK 99. (Topology) $\Gamma \approx \Gamma'$ means $\Gamma \backslash X$ and $\Gamma' \backslash X$ have a common finite cover.

LEMMA 100. *(Double coset decomposition) Let* $\Gamma \approx \Gamma'$ *and let* $\alpha \in \mathrm{Comm}(\Gamma)$. *Then* $\Gamma \backslash \Gamma\alpha\Gamma'$ *and* $\Gamma\alpha\Gamma'/\Gamma'$ *are finite.*

PROOF. Multiplying by α^{-1} on the left gives a bijection $\Gamma\backslash\Gamma\alpha\Gamma' \leftrightarrow (\alpha^{-1}\Gamma\alpha)\backslash\left(\alpha^{-1}\Gamma\alpha\Gamma'\right) \leftrightarrow \left(\alpha^{-1}\Gamma\alpha \cap \Gamma'\right)\backslash\Gamma'$. In fact, this shows that $\Gamma\alpha\Gamma' = \sqcup_i \Gamma\alpha_i$ where $\Gamma' = \sqcup_i \left(\alpha^{-1}\Gamma\alpha \cap \Gamma'\right)\alpha_i$. □

4.1.2. Abstract Hecke algebra.

DEFINITION 101. (Hecke algebra) Give a group G, a family Ξ of commensurable subgroups, and a semigroup Δ of G contained in the commensurator and containing all the subgroups. For $\Gamma, \Gamma' \in \Xi$ let $\mathscr{H}_R(\Gamma, \Gamma'; \Delta)$ be the free R-module generated by the double cosets $\{\Gamma\alpha\Gamma'\}_{\alpha\in\Delta}$. When $R = \mathbb{Z}$ we omit the subscript and we write $\mathscr{H}_R(\Gamma; \Delta)$ for $\mathscr{H}_R(\Gamma, \Gamma; \Delta)$.

DEFINITION 102. Given $\Gamma\alpha\Gamma' = \sqcup_i \Gamma\alpha_i$ and $\Gamma'\beta\Gamma'' = \sqcup_j \Gamma'\beta_j$ set $\Gamma\alpha\Gamma' \cdot \Gamma'\beta\Gamma'' = \sum_{i,j} \Gamma\alpha_i\beta_j\Gamma'' \in \mathscr{H}_R(\Gamma, \Gamma''; \Delta)$, and extend to a bilinear map $\mathscr{H}_R(\Gamma, \Gamma'; \Delta) \times \mathscr{H}_R(\Gamma', \Gamma''; \Delta) \to \mathscr{H}_R(\Gamma, \Gamma''; \Delta)$

LEMMA 103. $\Gamma\alpha\Gamma' \cdot \Gamma'\beta\Gamma''$ *is indep of the choice of representatives. The map is associative where defined.*

PROOF. The multiset of cosets $\Gamma\alpha_i\beta_j$ is Γ''-invariant. □

Now fix a right $R[\Delta]$-module M. For $m \in M^\Gamma$ and $\alpha \in \Delta$ write $m|\Gamma\alpha\Gamma' \stackrel{\text{def}}{=} \sum_i m|\alpha_i$ and extend linearly to $\mathscr{H}_R(\Gamma, \Gamma'; \Delta)$.

LEMMA 104. $m|\Gamma\alpha\Gamma' \in M^{\Gamma'}$ *and is indep of the choice of representatives. This extends to an action of the ensemble of operators.*

COROLLARY 105. *$\mathscr{H}_R(\Gamma;\Delta)$ is an associative algebra under the product with unit element Γ, and M^Γ is stable under this algebra.*

DEFINITION 106. $\deg(\Gamma\alpha\Gamma) = \#\Gamma\backslash\Gamma\alpha\Gamma$, extended linearly to $\mathscr{H}_R(\Gamma;\Delta)$ where it is multiplicative.

LEMMA 107. *If $\#\Gamma\backslash\Gamma\alpha\Gamma' = \#\Gamma\alpha\Gamma'/\Gamma'$ then they have a common set of reps.*

PROOF. $\Gamma\alpha\Gamma = \sqcup_i \Gamma\alpha_i$, α'_j on other side. if $\Gamma\alpha_i \subset \cup_{k\neq j}\alpha'_k\Gamma'$ then the same would hold for $\Gamma\alpha_i\Gamma'$, a contradiction, so $\Gamma\alpha_i \cap \alpha'_j\Gamma \neq \emptyset$. Now take $\delta_i \in \Gamma\alpha_i \cap \alpha'_i\Gamma'$. $\square$

THEOREM 108. *Let ι be an anti-involution of Δ fixing Γ such that $\Gamma\alpha\Gamma = \Gamma\alpha^\iota\Gamma$ for all $\alpha \in \Delta$. Then $\mathscr{H}_R(\Gamma;\Delta)$ is commutative.*

PROOF. Note first that the assumption means $(\Gamma\alpha\Gamma)^\iota = \Gamma\alpha\Gamma$ for all $\alpha \in \Delta$. In particular $\iota\colon \Gamma\backslash\Gamma\alpha\Gamma \to (\Gamma\alpha\Gamma)^\iota/\Gamma^\iota = \Gamma\alpha\Gamma/\Gamma$ is a bijection. By the Lemma we have $\Gamma\alpha\Gamma = \sqcup_i\Gamma\alpha_i = \sqcup_i\alpha_i\Gamma$ and $\Gamma\beta\Gamma = \sqcup_j\Gamma\beta_j = \sqcup_j\beta_j\Gamma$. Acting by ι we find $\Gamma = \sqcup_i\Gamma\alpha_i^\iota$ and $\Gamma = \sqcup_j\Gamma\beta_j^\iota$. We then have $\Gamma\alpha\Gamma\cdot\Gamma\beta\Gamma = \sum_{i,j}\Gamma\alpha_i\beta_j\Gamma$ and $\Gamma\beta\Gamma\cdot\Gamma\alpha\Gamma = \sum_{i,j}\Gamma\beta_j^\iota\alpha_i^\iota\Gamma$ and we are done. $\square$

On space of aut forms: equality of reps by volume computation. Adjoint of α is $(\det\alpha)\,\alpha^{-1}$.

4.1.3. Hecke operators. Let M by a right Δ-module. For $\Gamma,\Gamma' \in \Xi$ and $\alpha \in \Delta$ so that $\Gamma\backslash\Gamma\alpha\Gamma' = \sqcup_i\Gamma\alpha_i$, let $\Gamma\alpha\Gamma'\colon M^\Gamma \to M^{\Gamma'}$ be given by

$$m \mid \Gamma\alpha\Gamma = \sum_i m^{\alpha_i}\,.$$

LEMMA 109. *This is well-defined, associative where this makes sense.*

Now let $\Gamma \subset G = \mathrm{GL}_2(\mathbb{R})^+$ be a Fuchsian group, $\Gamma \subset \Delta \subset G = \mathrm{Comm}(\Gamma)$ a semigroup. Fix a character χ of Δ such that $\chi\restriction_\Gamma$ is of finite order. Let Ξ be the set of finite-index subgroups of Γ. Let Δ act on $\Omega_k(\Gamma,\chi)$ by

$$\begin{aligned} f^\alpha(z) &= \det(\alpha)^{k/2-1}\overline{\chi(\alpha)}\,(f|_k\alpha)(z) \\ &= \det(\alpha)^{k-1}\overline{\chi(\alpha)}\,j(\alpha,z)^{-k}f(\alpha z)\,. \end{aligned}$$

DEFINITION 110. For $\Gamma_1,\Gamma_2 \in \Xi$, $\alpha \in \Delta$ so that $\Gamma_1\alpha\Gamma_2 = \sqcup_i\Gamma_1\alpha_i$ we define the *Hecke operator* $\Gamma_1\alpha\Gamma_2\colon \Omega_k(\Gamma_1,\chi) \to \Omega_k(\Gamma_2,\chi)$ by

$$f|_k\Gamma_1\alpha\Gamma_2 = \sum_i f^{\alpha_i}\,.$$

LEMMA 111. *$\Gamma_1\alpha\Gamma_2$ preserves holomorphy in $\mathbb{H}$ and at the cusps, and also cuspdiality.*

PROPOSITION 112. *For $\alpha \in G$ set $\alpha' = \det(\alpha)\alpha^{-1}$. Let $f \in \mathcal{S}(\Gamma)$, $g \in \mathcal{H}(\Gamma)$. Then $(f|_k\alpha, g) = (f, g|_k\alpha)$ and $(f|_k\Gamma\alpha\Gamma, g) = (f, g|_k\Gamma\alpha'\Gamma)$.*

PROOF. Let $\Gamma' = \alpha^{-1}\Gamma\alpha \cap \Gamma$, think of $f|_k\alpha$ and g as elements of $\mathcal{H}_k(\Gamma')$, and evaluate $(f|_k\alpha, g)$ there. We then act by α^{-1} which is measure-preserving, letting $\Gamma'' = \Gamma \cap \alpha\Gamma\alpha^{-1}$ and noting that

$\alpha' z = \alpha^{-1} z$:

$$\begin{aligned}
(f|_k\alpha, g) &= \frac{1}{\mathrm{vol}\,(\Gamma'\backslash\mathbb{H})}\int_{\Gamma'\backslash\mathbb{H}} \det(\alpha)^{k/2}(cz+d)^{-k} f(\alpha z)\overline{g(z)}y(z)^k dA(z) \\
&= \frac{1}{\mathrm{vol}\,(\Gamma''\backslash\mathbb{H})}\int_{\Gamma''\backslash\mathbb{H}} \det(\alpha)^{k/2}(c(\alpha^{-1}z)+d)^{-k} f(z)\overline{g(\alpha^{-1}z)}y(\alpha^{-1}z)^k dA(z) \\
&= \frac{1}{\mathrm{vol}\,(\Gamma''\backslash\mathbb{H})}\int_{\Gamma''\backslash\mathbb{H}} \det(\alpha')^{k/2}(c(\alpha^{'}z)+d)^{-k} f(z)\overline{g(\alpha' z)}y(\alpha^{'}z)^k dA(z) \\
&= \frac{1}{\mathrm{vol}\,(\Gamma''\backslash\mathbb{H})}\int_{\Gamma''\backslash\mathbb{H}} \det(\alpha')^{k/2}(c\frac{dz-b}{-cz+a}+d)^{-k} f(z)\overline{g(\alpha' z)}\frac{\det(\alpha')^k y(z)^k}{|-cz+a|^{2k}} dA(z) \\
&= \frac{1}{\mathrm{vol}\,(\Gamma''\backslash\mathbb{H})}\int_{\Gamma''\backslash\mathbb{H}} \det(\alpha')^{k/2}\frac{\det(\alpha')^k}{(cdz-bc-dcz+ad)^{-k}} f(z)\overline{(-cz+a)^{-k}g(\alpha' z)}y(z)^k dA(z) \\
&= \frac{1}{\mathrm{vol}\,(\Gamma''\backslash\mathbb{H})}\int_{\Gamma''\backslash\mathbb{H}} f(z)\overline{(g|_k\alpha')\,(z)}y(z)^k dA(z) \\
&= (f, g|_k\alpha)\,.
\end{aligned}$$

Moreover, $\mathrm{vol}(\Gamma'\backslash\mathbb{H}) = \mathrm{vol}(\Gamma''\backslash\mathbb{H})$ shows $[\Gamma:\Gamma'] = [\Gamma:\Gamma'']$ and hence $\Gamma\backslash\Gamma\alpha\Gamma$ and $\Gamma\alpha\Gamma/\Gamma$ have a common set of representatives, say α_i. It follows that $\Gamma\alpha\Gamma = \sqcup_i\Gamma\alpha_i$ and $\Gamma\alpha^{-1}\Gamma = \sqcup_i\Gamma\alpha_i^{-1}$ and hence $\Gamma\alpha^{'}\Gamma = \sqcup_i\Gamma\alpha_i^{'}$. Thus:

$$\begin{aligned}
(f|\Gamma\alpha\Gamma, g) &= \sum_i \det(\alpha)^{k/2-1}(f|_k\alpha_i, g) \\
&= \sum_i \det(\alpha')^{k/2-1}\left(f, g|_k\alpha_i^{'}\right) \\
&= \left(f, g|\Gamma\alpha'\Gamma\right)\,.
\end{aligned}$$

□

COROLLARY 113. *The Hecke algebra $\mathscr{H}_R(\Gamma;\Delta)$ preserves the spaces of cusp forms and Eisenstein series; spaces corresponding to different character are orthogonal.*

4.2. Congruence subgroups and their Hecke algebras

DEFINITION 114. $\Gamma < \mathrm{SL}_2(\mathbb{Z})$ is called a *congruence subgroup* if $\Gamma \supset \Gamma(N)$ for some N.

REMARK 115. $\Gamma(N) \subset \Gamma_1(N) \subset \Gamma_0(N)$. Also $\begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}^{-1}\Gamma(N)\begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix} \supset \Gamma_0(N^2)$. It follows it's enough to consider one of the subgroups.

Now $\Gamma_1(N) \lhd \Gamma_0(N)$ with quotient $(\mathbb{Z}/N\mathbb{Z})^\times$. For χ a Dirichlet character mod N and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ set $\chi(\gamma) = \chi(d)$, and write:

$$\Omega_k(N,\chi) = \Omega_k(\Gamma_0(N),\chi)\,.$$

REMARK 116. This is an orthogonal decomposition of $\Omega_k(\Gamma_1(N))$ (consider Hecke operators given by $\gamma \in \Gamma_0(N)$).

LEMMA 117. *(standard involutions)*

(1) *Let* $\omega_N = \begin{pmatrix} & -1 \\ N & \end{pmatrix}$. *Then* $\omega_N^2 = \begin{pmatrix} -N & 0 \\ 0 & -N \end{pmatrix}$, *and* $f \mapsto f|_k \omega_N$ *gives isomorphisms* $(N,\chi) \to (N,\bar{\chi})$.
(2) *Let* $\bar{f}(z) = \overline{f(-\bar{z})}$. *This is also such an isom; in terms of the Fourier expansion this maps* $\sum_{n\geq 0} a_n e(nz)$ *to* $\sum_{n\geq 0} \bar{a}_n e(nz)$.

PROOF. First part clear. For second part use the following involution on $\mathrm{SL}_2(\mathbb{Z})$: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$. □

DEFINITION 118. $\Delta_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z})^+ \mid c \equiv 0(N),\, (a,N) = 1 \right\}$ and $\Delta_0^*(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}$

REMARK 119. The two involutions carry one to the other.

LEMMA 120. *These are semigroups containing* $\Gamma_0(N)$*; for* $\alpha \in \Delta_0(N)$ *there are unique* $l|m$ *positive,* $(l,N) = 1$ *such that* $\Gamma_0(N)\alpha\Gamma_0(N) = \Gamma_0(N) \begin{pmatrix} l & 0 \\ 0 & m \end{pmatrix} \Gamma_0(N)$.

PROOF. $\Lambda = \mathbb{Z}^2$, $\Lambda_0 = \mathbb{Z}e_1 \oplus N\mathbb{Z}e_2$. Then $\alpha\Lambda_0 \subset \Lambda_0$ and $[L : \alpha L_0] = [L : \alpha L]\,[L : L_0] = nN$ with $n = \det(\alpha)$. We have $\alpha L_0 = \mathbb{Z}aw_1 \oplus \mathbb{Z}bw_2$ for a basis w_1, w_2 of L (we may assume it positively orientied by changing w_2 to $-w_2$). Here $a|b$ and $ab = nN$. We have $(a,N) = 1$ since $\alpha L_0 \not\subset tL$ for any divisor t (check first co-ordinate). Let $l = a$, $m = b/N$. Then $L_0 = \mathbb{Z}w_1 \oplus \mathbb{Z}Nw_2$ (unique submodule of index N contaning αL_0) and $\alpha L = \mathbb{Z}lw_1 \oplus \mathbb{Z}mw_2$ since this is unique submodule of L of index n containing αL_0. Change of basis and we are now done. $L/\alpha L$ gives l, m uniquely. □

COROLLARY 121. *If* $(lm,N) = 1$ *then* $\Gamma_0(N) \begin{pmatrix} l & 0 \\ 0 & m \end{pmatrix} \Gamma_0(N) = \Gamma_0(N) \begin{pmatrix} m & 0 \\ 0 & l \end{pmatrix} \Gamma_0(N)$.

THEOREM 122. *The Hecke algebras are commutative.*

PROOF. Use involution $\begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \mapsto \begin{pmatrix} a & c \\ Nb & d \end{pmatrix}$. □

DEFINITION 123. For $l|m$, $(l,N) = 1$ set $T(l,m) = \Gamma_0(N) \begin{pmatrix} l & 0 \\ 0 & m \end{pmatrix} \Gamma_0(N)$, for $(n,N) = 1$ set $T_n = \sum_{\det(\alpha)=n} \Gamma_0(N)\alpha\Gamma_0(N) = \sum_{lm=n} T(l,m)$. For p prime, $T_p = T(1,p)$. Stars indicate operators wrt $\Delta_0^*(N)$ (put superscript (N) to indicate the level if needed).

For a Dirichlet character χ mod N set $\chi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \bar{\chi}(a)$ on $\Delta_0(N)$ and $\chi^*\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \chi(d)$ on $\Delta_0^*(N)$. Both restrict to the usual character of $\Gamma_0(N)$.

REMARK 124. Conjugation by ω_N maps $\Delta_0(N)$ to $\Delta_0^*(N)$ and χ^* to $\bar{\chi}$.

PROPOSITION 125. *For* $f \in \mathcal{H}_k(N,\chi)$ *and* $n = lm$ *relatively prime to* N
(1) $f|T^*(m,l) = \bar{\chi}(lm) \cdot f|T(l,m)$ *and* $f|T_n^* = \bar{\chi}(n) \cdot f|T_n$. *In particular,* $T(l,m)$ *and* $T^*(m,l)$ *commute as do* T_n *and* T_n^*.
(2) *The adjoint of* $T(l,m)$ *is* $T^*(m,l)$*, the same for* T_n, T_n^*.

PROOF. First, say $\Gamma_0(N)\begin{pmatrix} l & 0 \\ 0 & m \end{pmatrix}\Gamma_0(N) = \Gamma_0(N)\begin{pmatrix} m & 0 \\ 0 & l \end{pmatrix}\Gamma_0(N) = \sqcup_i \Gamma_0(N)\alpha_i$. Also, if $\alpha \in \Delta_0(N) \cap \Delta_0^*(N)$ then $\chi^*(\alpha) = \chi(a) = \bar{\chi}(d)\chi(ad) = \chi(\det(\alpha))\chi(\alpha)$, so

$$\begin{aligned} f|T^*(m,l) &= \sum_i \overline{\chi^*(\alpha_i)} f^{\alpha_i} \\ &= \bar{\chi}(lm)\sum_i \overline{\chi(\alpha_i)} f^{\alpha_i} \\ &= f|T(l,m). \end{aligned}$$

Next, by Prop 112 the adjoint of $T(l,m)$ is $\Gamma_0(N)\begin{pmatrix} l & 0 \\ 0 & m \end{pmatrix}'\Gamma_0(N) = \Gamma_0(N)\begin{pmatrix} m & 0 \\ 0 & l \end{pmatrix}(N) = T^*(m,l)$. □

COROLLARY 126. *All the $T(l,m)$ and T_n (prime to N) can be jointly diagonalized.*

PROOF. This is a commuting family of normal operators. □

REMARK 127. The action of ω_N conjugates $T(l,m)$ acting on $\mathcal{H}_k(N,\chi)$ to the action of $T^*(m,l)$ on $\mathcal{H}_k(N,\bar{\chi})$.

LEMMA 128. *For $e \geq 1$, $\Gamma_0(N)\backslash\Gamma_0(N)\begin{pmatrix} 1 & 0 \\ 0 & p^e \end{pmatrix}\Gamma_0(N)$ is given by $\begin{pmatrix} p^{e-f} & m \\ 0 & p^f \end{pmatrix}$ where $0 \leq f \leq e$, $0 \leq m < p^f$, $(p^f, p^{e-f}, m) = 1$ if $p \nmid N$, only $f = e$ if $p|N$.*

PROOF. Let $\beta = \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \in \Gamma_0(N)\begin{pmatrix} 1 & 0 \\ 0 & p^e \end{pmatrix}\Gamma_0(N)$. Then $\det(\beta) = p^e$ so $(a,Nc) = (a,c)$ is a power of p, (equality since $(a,N) = 1$). If $p|N$ then $(a,Nc) = 1$ and so there is $\gamma_1 = \begin{pmatrix} * & * \\ -Nc & a \end{pmatrix}$. Then $\gamma_1\beta = \begin{pmatrix} 1 & * \\ 0 & p^e \end{pmatrix}$. If $(p,N) = 1$ set $a' = a/(a,c)$ and $c' = c/(a,c)$. These are relatively prime so can choose $\gamma_1 == \begin{pmatrix} * & * \\ -Nc' & a' \end{pmatrix}$ then $\gamma_1\beta = \begin{pmatrix} p^{e-f} & * \\ 0 & p^f \end{pmatrix}$. In any case now act by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ on the left to make $0 \leq m < p^f$. It follows that the double coset is contained by the cosets of $\begin{pmatrix} p^{e-f} & m \\ 0 & p^f \end{pmatrix}$ where $0 \leq f \leq e$, $0 \leq m < p^f$. We first show that these cosets are disjoint. Indeed, if

$$\gamma\begin{pmatrix} p^{e-f} & m \\ 0 & p^f \end{pmatrix} = \begin{pmatrix} p^{e-g} & n \\ 0 & p^g \end{pmatrix}.$$

Then γ is integral, upper-triangular with diagonal p^{f-g}, p^{g-f} so $f = g$ and $\gamma = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ so $m \equiv n\,(p^f)$. Finally, we checke which of these cosets lie in the double coset: we have

$$\begin{pmatrix} 1 & m \\ 0 & p^e \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & p^e \end{pmatrix}\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \in \Gamma_0(N)\begin{pmatrix} 1 & 0 \\ 0 & p^e \end{pmatrix}\Gamma_0(N)$$

and

$$\Gamma_0(N)\begin{pmatrix} p^{e-f} & m \\ 0 & p^f \end{pmatrix}\Gamma_0(N) = \Gamma_0(N)\begin{pmatrix} p^a & 0 \\ 0 & p^b \end{pmatrix}\Gamma_0(N)$$

for some $0 \le a \le b$ with $a+b=e$. We consider the quotient $\mathbb{Z}^2/\alpha\mathbb{Z}^2$. This is $\mathbb{Z}/p^a\mathbb{Z} \oplus \mathbb{Z}/p^b\mathbb{Z}$ so the double coset is from $\begin{pmatrix} 1 & 0 \\ 0 & p^e \end{pmatrix}$ iff $\alpha\mathbb{Z}^2 \not\subset p\mathbb{Z}^2$, which is the case iff $\left(p^{e-f}, p^f, m\right) = 1$. □

COROLLARY 129. $\deg\left(\Gamma_0(N)\begin{pmatrix} 1 & 0 \\ 0 & p^e \end{pmatrix}\Gamma_0(N)\right) = 1 + \sum_{f=1}^{e-1}(p^f - p^{f-1}) + p^e = p^e + p^{e-1}$.

EXERCISE 130. If $(m,m')=1$ then $T(l,m)T(l',m') = T(ll',mm')$; if $(n,n')=1$ then $T_nT_{n'} = T_{nn'}$. Also, $T_n = \sqcup_{(a,N)=1} \sqcup_{ad=n} \sqcup_{0\le b<d} \Gamma_0(N)\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$. This the Hecke algebra is the polynomial ring on $T_p, T(p,p)$ for $p \nmid N$ and T_q for $q|N$.

LEMMA 131. *Let $\gamma_a \in \Gamma_0(N)$ satisfy $\gamma_a \equiv \begin{pmatrix} 1 & a \\ & 1 \end{pmatrix}$ (p), and if $p \nmid N$ let $\gamma_p \in \Gamma(N)$ satisfy $\gamma_p \equiv \begin{pmatrix} 0 & -a \\ a^{-1} & 0 \end{pmatrix}$ (p). Then representatives for $\Gamma_0(pN)\begin{pmatrix} 1 & \\ & p \end{pmatrix}\Gamma_0(N)$ are given by $\left\{\Gamma_0(pN)\begin{pmatrix} 1 & \\ & p \end{pmatrix}\gamma_a\right\}_{a(p)}$ plus $\Gamma_0(pN)\begin{pmatrix} 1 & \\ & p \end{pmatrix}\gamma_p$.*

PROOF. Let $\Gamma' = \Gamma_0(N) \cap \begin{pmatrix} 1 & \\ & p \end{pmatrix}^{-1}\Gamma_0(pN)\begin{pmatrix} 1 & \\ & p \end{pmatrix} = \Gamma_0(N) \cap \Gamma^0(p)$ (transpose). Need representatives for $\Gamma' \backslash \Gamma_0(N)$. Write $N = N'p^e$ with $(p,N')=1$, reduce mod pN and use the Chinese Remainder Theorem. In the $\mathrm{SL}_2(\mathbb{Z}/N'\mathbb{Z})$ factor both map to the same subgroup, so it remains to consider the other factor. If $N = N'$ then $\Gamma_0(N)$ surjects there while Γ' maps to lower-triangular matrices with the quotient of size $p+1$. Otherwise a direct calculation finishes the proof. □

4.3. Fourier expansion and Newforms (Atkin-Lehner Theory)

4.3.1. The space of newforms. Note that $i\infty$ is a cusp of $\Gamma_0(N)$ of width 1. Let

$$f(z) = \sum_{m=0}^{\infty} c_m e(mz).$$

Then

$$\begin{aligned}
(f|T_n)(z) &= n^{k-1}\sum_{ad=n}\sum_{b=0}^{d-1}\chi(a)d^{-k}f((az+b)/d) \\
&= n^{k-1}\sum_{ad=n}\chi(a)d^{-k}\sum_{m=0}^{\infty}\sum_{b=0}^{d-1}c_m e\left(\frac{maz+mb}{d}\right) \\
&= n^{k-1}\sum_{ad=n}\chi(a)d^{-k}d\sum_{m=0}^{\infty}c_{md}e(maz) \\
&= \sum_{m=0}^{\infty}\left(\sum_{0<a|(m,n)}\chi(a)a^{k-1}c_{\frac{mn}{a^2}}\right)e(mz).
\end{aligned}$$

In particular, the 1st coefficient is c_n.

COROLLARY 132. *(Weak multiplicity one) Assume $f|T_n = \lambda_n f$ for all n. Then $c_n = \lambda_n c_1$, and in particular f is uniquely determined up to a constant multiple.*

Now fix $\delta_l = \begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix} \in \Delta_0^*(N)$, so $f(lz) = l^{-k/2}\left(f|_k\delta_l\right)(z)$. Since $\delta_l\Gamma_0(lN)\delta_l^{-1} \subset \Gamma_0(N)$ it follows that $f \in \mathcal{H}_k(N,\chi) \Rightarrow f(lz) \in \mathcal{H}_k(lN,\chi)$, and that this preserves cuspidality (can interpret this via the Hecke operator $\Gamma_0(N)\delta_l\Gamma_0(lN) = \Gamma_0(N)\delta_l$).

DEFINITION 133. Let m_χ be the conductor of χ, and let $\mathcal{S}_k^{\text{old}}(N,\chi) = \sum_{m_\chi|M\neq N}\sum_{Ml|N}\mathcal{S}_k(M,\chi)\cdot\delta_l \subset \mathcal{S}_k(N,\chi)$. Call these *oldforms*. The orthogonal complement $\mathcal{S}_k^{\text{new}}(N,\chi)$ is called the *space of newforms*.

EXAMPLE 134. If χ is primitive then $\mathcal{S}_k(N,\chi) = \mathcal{S}_k^{\text{new}}(N,\chi)$.

REMARK 135. It is clear that $\mathcal{S}_k^{\text{old}}(N,\chi)$ is generated by the images of $\mathcal{S}_k^{\text{new}}(M,\chi)$ under δ_l.

4.3.2. "Level lowering".

LEMMA 136. *If $(n,lM)=1$ then T_n commutes with the two embeddings $\mathcal{H}_k(M,\chi)\to\mathcal{H}_k(lM,\chi)$ given by the identity map and δ_l. More precisely, for $f\in\mathcal{H}_k(M,\chi)$,*

(1) $f|T_n^{(M)} = f|T_n^{(lM)}$.
(2) $f|T_n^{(M)}\cdot\Gamma_0(M)\delta_l\Gamma_0(lM) = f|\Gamma_0(M)\delta_l\Gamma_0(lM)\cdot T_n^{(lM)}$

PROOF. Enough to check for $T(p,p)$ and $T(p)$, and the action of $T(p,p)$ is always the same. For $T(p)$ in case (1) the coset representatives are the same (Lemma 128). In case (2), we take the standard representatives $\left\{\begin{pmatrix} 1 & b \\ & p \end{pmatrix}\right\}_{b(p)} \cup \left\{\begin{pmatrix} p & \\ & 1 \end{pmatrix}\right\}$. Then

$$\begin{aligned} f|_k\delta_l|T_p^{(lM)} &= p^{k/2-1}\sum_{b(p)} f|_k\delta_l\begin{pmatrix} 1 & b \\ & p \end{pmatrix} + p^{k/2-1}\chi(p)f|_k\delta_l\begin{pmatrix} p & \\ & 1 \end{pmatrix} \\ &= p^{k/2-1}\sum_{b(p)} f|_k\begin{pmatrix} 1 & lb \\ & p \end{pmatrix}\delta_l + p^{k/2-1}\chi(p)f|_k\begin{pmatrix} p & \\ & 1 \end{pmatrix}\delta_l \\ &= f|T_p^{(M)}\delta_l \end{aligned}$$

since $(l,p)=1$. $\square$

COROLLARY 137. *If $(n,N)=1$ then the action of $T_n^{(N)}$ preserves the decomposition $\mathcal{S}_k(N,\chi) = \mathcal{S}_k^{old}(N,\chi)\oplus\mathcal{S}_k^{new}(N,\chi)$.*

PROOF. The adjoint of T_n is $\bar{\chi}(n)T_n$. $\square$

LEMMA 138. *If $p\nmid l$ then the two embeddings intertwine $\Gamma_0(pN)\begin{pmatrix} 1 & \\ & p \end{pmatrix}\Gamma_0(N)$ and $\Gamma_0(pM)\begin{pmatrix} 1 & \\ & p \end{pmatrix}\Gamma_0($ $M=lN$.*

PROOF. Similar argument using the representatives of Lemma 131 and noting that $p|M$ iff $p|N$. $\square$

THEOREM 139. *Let $f=\sum_{n\geq 0}a_n e(nz)\in\mathcal{H}_k(N,\chi)$ and assume $a_n=0$ for all n prime to l. Then for each prime $p|\left(l,N/m_\chi\right)$ there is $f_p\in\mathcal{H}_k(N/p,\chi)$ such that*

$$f(z)=\sum_p f_p(pz)\,.$$

Moreover, if f is a cusp form we may choose the f_p to be cusp forms as well, and in particular $f \in SkoNx$.

We may assume l squarefree, and will proceed by induction on the number of prime factors. When l is prime, we note that $f(z/l)$ satisfies the assumptions of the following theorem:

PROPOSITION 140. *Let l be a positive integer, and let f be a function on $T^{\mathbb{Z}}\backslash\mathbb{H}$ such that $f(lz) \in \mathcal{H}_k(N,\chi)$.If $lm_\chi|N$ then $f \in \mathcal{H}_k(N/l,\chi)$; otherwise $f = 0$. If $f(lz)$ is cusp form then so is f.*

PROOF. By induction may assume l prime. Let $\Gamma' = \Gamma_0(N) \cap \Gamma^0(l)$ (transpose). Then $\delta_l^{-1}\Gamma'\delta_l \subset \Gamma_0(lN)$. For $\gamma \in \Gamma'$ we have:

$$\begin{aligned} f|_k\gamma &= f|_k\delta_l\delta_l^{-1}\gamma\delta_l\delta_l^{-1} \\ &= (f|_k\delta_l)\,|_k\delta_l^{-1}\gamma\delta_l|_k\delta_l^{-1} \\ &= \chi(\delta_l^{-1}\gamma\delta_l)f|_k\delta_l|_k\delta_l^{-1} \\ &= \chi(\gamma)f\,. \end{aligned}$$

It follows that $f \in \mathcal{H}_k(\Gamma'',\chi)$ where Γ'' is the group generated by Γ' and T. $[\Gamma'':\Gamma'] \geq l$ while $[\Gamma_0(N):\Gamma'] \leq \left[\Gamma(1):\Gamma^0(l)\right] = l+1$ so $\Gamma'' = \Gamma_0(N)$ and $f, f|_k\delta_l \in \mathcal{S}_k(N,\chi)$.

Assume first that $(l,N) = 1$. Then $\Gamma_0(N)\delta_l\Gamma_0(N) = \Gamma_0(N)\begin{pmatrix} 1 & \\ & l \end{pmatrix}\Gamma_0(N)$. Since $\begin{pmatrix} 1 & \\ & l \end{pmatrix} T \begin{pmatrix} 1 & \\ & l \end{pmatrix}^{-1} \notin \Gamma_0(N)$, there is $\gamma \in \Gamma_0(N)$ such that $\delta_l\gamma\delta_l^{-1} \notin \Gamma_0(N)$. But $\delta_l\gamma\delta_l' \in \Delta_0(N)$ so there are $0 < u|v$ and $\gamma_1, \gamma_2 \in \Gamma_0(N)$ so that $\gamma_1\delta_l\gamma\delta_l'\gamma_2 = \begin{pmatrix} u & \\ & v \end{pmatrix}$. We have $u < v$ (otherwise $\delta_l\gamma\delta_l^{-1} \in \Gamma_0(N)$) and then $f|_k\begin{pmatrix} u & \\ & v \end{pmatrix} = Cf$ for $C \neq 0$ so $f(z/h) = C'f$ for $C' \neq 0$ for $h = v/u > 1$. The Fourier expansion shows that this is impossible, so $f = 0$.

Assume next that $l|N$. Then $f|_k\begin{pmatrix} 1 & \\ N/l & 1 \end{pmatrix} = f|_k\delta_l\begin{pmatrix} 1 & \\ N & 1 \end{pmatrix}\delta_l^{-1} = f$. Since $\Gamma_1(l/N)$ is generated by this element and $\Gamma_1(N)$ we can write f uniquely in the form $\sum_\rho f_\rho$ where $f_\rho \in \mathcal{S}_k(\frac{N}{l},\rho)$. Now let $\gamma \in \Gamma_0(N)$. Then $\sum_\rho \rho(\gamma)f_\rho = f|_k\gamma = \chi(\gamma)f$. If $f \neq 0$ it follows that χ extends a character ρ mod N/l, and in particular that $m_\chi|\frac{N}{l}$. □

We note that the argument in the case $(l,N) = 1$ shows:

LEMMA 141. *(Hecke) If $\alpha \in \Delta_0(N)$ with $\det(\alpha) > 1$ and prime to N has relatively prime entries, then $f, f|_k\alpha \in \mathcal{H}_k(N,\chi)$ implies $f = 0$.*

Returning to the proof of Theorem 139, we start with a weaker form of the Theorem.

PROPOSITION 142. *Let $l > 1$ be squarefree, $f \in \mathcal{H}_k(N,\chi)$ with $a_n = 0$ if $(n,l) = 1$. Then $f = \sum_{p|l} g_p(pz)$ with $g_p \in \mathcal{H}_k(Nl^2,\chi)$ (if $l|N$, level Nl suffices). If f is cuspidal so are the g_p.*

PROOF. By induction on the number of prime factors. If l is prime then $g(z) = \sum a_n e(\frac{nz}{l})$ satisfies the assumptions of Prop 140 and so if $f \neq 0$ we have $f = g(lz)$ with $g \in \mathcal{H}_k(N/l,\chi) \subset \mathcal{H}_k(Nl^2,\chi)$. Otherwise fix a prime factor p of l and write

$$g_p(z) = \sum_{p|n} a_n e\left(\frac{n}{p}z\right).$$

Then $g_p(z) \in \mathcal{H}_k(Np,\chi)$. Indeed, let $N' = \mathrm{lcm}(N,p)$. Then $f|T_p^{(N')}(z) \propto \sum_{n\geq 0} a_n \sum_{b(p)} e(n\frac{z+b}{p}) = \sum_{p|n} a_n e(\frac{n}{p}z)$ so $g_p(z) \in \mathcal{H}_k(N',\chi)$ and is a cusp form if f is. Consider now

$$f_1 = f(z) - g_p(pz)\,.$$

This belongs to $\mathcal{H}_k(Np^2,\chi)$ (level Np if $p|N$, cuspidal if f is) and is supported on Fourier coefficients which not coprime to l by comprime to p, hence not comprime to $l' = \frac{l}{p}$. By induction

$$f_1 = \sum_{p'|l'} g_{p'}(p'z)$$

with $g_{p'} \in \mathcal{H}_k(Np^2(l')^2,\chi)$ and we are done. □

In fact, we have also shown:

PROPOSITION 143. *("pre-twisting") let $f \in \mathcal{H}_k(N,\chi)$ and let l be squarefree. Then $\sum_{(n,l)=1} a_n e(nz) \in \mathcal{H}_k(Nl^2,\chi)$, cuspidal if f is (more precisely, for $p|l$ get a factor p if $p|N$, p^2 otherwise).*

PROOF. The same argument as above shows that $\sum_{p|n} a_n e(nz) \in \mathcal{H}_k(Np^e,\chi)$ with $e \in \{1,2\}$ as required. Subtract this from f and continue by induction. □

We can now complete the induction step of Theorem 139.

PROOF. Let f be such that $a_n = 0$ if $(n,l) = 0$. Fix a prime factor p of l, set $l' = l/p$ and set

$$\begin{aligned} g(z) &= \sum_{(n,l')=1} a_n e(nz) \\ h(z) &= \sum_{(n,l')\neq 1} a_n e(nz) \end{aligned}$$

so that $f = g+h$. We have $g \in \mathcal{H}_k(Nl^2,\chi)$ and its Fourier coefficients are non-zero at multiples of p. If $pm_\chi \nmid N$ then $pm_\chi \nmid Nl'^2$ so $g = 0$ (Prop 140), $f = h$ and we are done by induction. Otherwise put $g_p = g(\frac{z}{p}) \in \mathcal{H}_k(\frac{N}{p}l'^2,\chi)$. A calculation with the explicit representatives (Lemma 131) shows that

$$g\Big|\Gamma_0(Nl'^2)\begin{pmatrix} 1 & \\ & p \end{pmatrix}\Gamma_0(Nl'^2/p) = \frac{d}{p}g_p$$

where d is the degree of the Hecke operator. Thus

$$g(z) = g_p(pz) = \frac{p}{d}\left(g\Big|\Gamma_0(Nl'^2)\begin{pmatrix} 1 & \\ & p \end{pmatrix}\Gamma_0(Nl'^2/p)\right)(pz)$$

and we set

$$f_p(z) = \frac{p}{d}\left(f\Big|\Gamma_0(N)\begin{pmatrix} 1 & \\ & p \end{pmatrix}\Gamma_0(N/p)\right)(z)\,.$$

Then $f_p \in \mathcal{H}_k(N/p,\chi)$. Finally, consider $f(z) - f_p(pz) \in HkNx$. By the intertwining property,

$$f_p(z) = \frac{p}{d}\left(f\Big|\Gamma_0(Nl'^2)\begin{pmatrix} 1 & \\ & p \end{pmatrix}\Gamma_0(Nl'^2/p)\right)(z)$$

and hence

$$\begin{aligned} f(z) - f_p(pz) &= (f-g)(z) - (f_p - g_p)(pz) \\ &= h(z) - \frac{p}{d}\left(h \Big| \Gamma_0(Nl'^2) \begin{pmatrix} 1 & \\ & p \end{pmatrix} \Gamma_0(Nl'^2/p)\right)(pz). \end{aligned}$$

Since the FC of h are supported on integers no coprime to l',

$$h(z) = \sum_{q|l'} h_q(qz)$$

with $h_q \in \mathcal{H}_k(Nl'^3, \chi)$ and since $p|N$ we also have

$$h \Big| \Gamma_0(l'^2 N) \begin{pmatrix} 1 & \\ & p \end{pmatrix} \Gamma_0(Nl'^2/p) = h \Big| \Gamma_0(l'^3 qN) \begin{pmatrix} 1 & \\ & p \end{pmatrix} \Gamma_0(Nl'^3 q/p)$$

and since δ_q intertwines this Hecke operator,

$$h \Big| \Gamma_0(l'^2 N) \begin{pmatrix} 1 & \\ & p \end{pmatrix} \Gamma_0(Nl'^2/p) = \sum_{q|l'} \left(h_q \Big| \Gamma_0(l'^3 qN) \begin{pmatrix} 1 & \\ & p \end{pmatrix} \Gamma_0(Nl'^3 q/p)\right)(qz)$$

Now every Fourier coefficient of this expression has $q|n$ for some $q|l'$, and the same holds for h so $f(z) - f_p(pz)$ has all its Fourier coefficients with $(n, l') = 1$ vanish, and we may continue the induction. $\square$

4.3.3. Newforms. The key result of the theory is the following:

COROLLARY 144. *Let $f \in \mathcal{S}_k^{new}(N, \chi)$ be an eigenfunction of T_n, $(n, L) = 1$. Then $a_1(f) \neq 0$.*

PROOF. The explicit calcluation above shows that $a_1(f) = 0$ forces $a_n(f) = 0$ for $(n, L) = 1$, and the Theorem shows f is then an oldform. $\square$

THEOREM 145. *(Stronger multiplicity one) Let $f, g \in \mathcal{S}_k(N, \chi)$ with $f \in \mathcal{S}_k^{new}(N, \chi)$ non-zero, and assume for each $(n, L) = 1$ we have $f|T_n = \lambda_n f$ and $g|T_n = \lambda_n g$. Then $g \propto f$.*

PROOF. May assume $a_1(f) = 1$ and $N|L$. Let $g = g^{\mathrm{old}} + g^{\mathrm{new}}$. Then each is a common eigenfunction, and $g^{\mathrm{new}} - a_1(g^{\mathrm{new}}) f$ is supported on $(n, L) = 1$ so is an oldform hence vanishes. We may thus assume $g \in \mathcal{S}_k^{\mathrm{old}}(N, \chi)$ and show $g = 0$. For this write $g(z) = \sum_v g_v(l_v z)$ with $g_v \in \mathcal{S}_k^{\mathrm{new}}(M_v, \chi)$ with $l_v M_v | N$ and $M_v < N$. We may assume each g_v is an eigenfunction of T_n ($(n, L) = 1$) with same eigenvalues as f, g. In particular, $a_1(g_1) \neq 0$. Consider $g_1 - a_1(g_1) \cdot f$. Considering the Hecke eigenvalues the Fourier coefficients prime to L vanish so this is an oldform. But then f is an oldform, a contradiction. $\square$

DEFINITION 146. A *newform* (or *primitive form*) *of conductor N* is a function $f \in \mathcal{S}_k^{\mathrm{new}}(N, \chi)$such that:

(1) f is a common eigenfunction of T_n, $(n, N) = 1$.
(2) $a_1(f) = 1$ (“arithmetic normalization”)

THEOREM 147. *The newforms are common eigenfunctions of the whole Hecke algebra, and consitute a basis of SknNx.*

PROOF. Let f be a newform. Then $f|T_k$ has same T_n-eigenvalues for all $(n,N)=1$. By the stronger multiplicity one theorem it follows that $f|T_k = \lambda_k(f)\cdot f$. Next, the T_n may be jointly diagonalized in *SknNx*. Normalizing the elements of any basis gives a set of newforms, and every newform must be represented there. $\square$

COROLLARY 148. *Let $f \in \mathcal{S}_k(N,\chi)$ be a joint eigenfunction of T_n. Then there is $m_\chi | M | N$ and a newform g of conductor M with same T_n-eigenvalues. If f isn't in the space of newforms then $M < N$.*

4.3.4. The involutions. f a newform of conductor N, character χ. Then $f|T_n = a_n(f)\cdot f$ (Hecke ev = Fourier coeff). If $(n,N)=1$ then T_n^* is the adjoint so

$$\bar{a}_n = \bar{\chi}(n) a_n .$$

LEMMA 149. *The two involutions ω_N and $f \mapsto \bar{f}$ preserve the decomposition into oldforms and newforms (switching between χ, $\bar{\chi}$). $f \mapsto \bar{f}$ maps newforms to newforms.*

Since $f|_k\omega_N$ and $\bar{f}$ have the same Hecke ev $f|_k\omega_N = c\bar{f}$ for $c \in \mathbb{C}^\times$.

PROPOSITION 150. *("Twisting") Let $f(z) = \sum_{n\geq 0} a_n e(nz) \in \mathcal{H}_k(N,\chi)$ and let ψ be a Dirichlet characater mod q. Then*

$$g(z) = \sum_{n=0}^{\infty} a_n \psi(n) e(nz) \in \mathcal{H}_k(M, \chi\psi)$$

where $M = N \cdot \prod_{p|(q,N)} p \prod_{p|\frac{q}{(q,N)}} p^2$.

DEFINITION 151. Write $g = f \otimes \psi$.

4.4. Number theory

Integrality of the Fourier expansion, number field of definition.

CHAPTER 5

L-functions and the Converse Theorem

Math 613: Problem set 6 (due xx/11/09)

Dirichlet characters

Let $N \geq 1$. A *Dirichlet character mod* N is a non-zero function $\chi \colon \mathbb{Z} \to \mathbb{C}$ such that $\chi(ab) = \chi(a)\chi(b)$, $\chi(a) = \chi(b)$ if $a \equiv b\,(N)$ and $\chi(a) = 0$ whenever $(a,N) > 1$. We freely identify χ with the function it induces on $\mathbb{Z}/N\mathbb{Z}$

1. Let χ be a Dirichlet character mod N.
   (a) Show that $\chi(1) = 1$ and that χ is non-zero in $(\mathbb{Z}/N\mathbb{Z})^{\times}$.
   (b) Show that the non-zero values taken by χ are roots of unity.
   (c) Let ψ, χ be two Dirichlet characters mod N. Show that $\psi\chi$ is also such a character.

2. (The conductor) Let χ be a Dirichlet character mod N.
   (a) Let $N|M$. Show that the function $\chi_M(a) = \begin{cases} \chi(a) & (a,M) = 1 \\ 0 & (a,M) > 1 \end{cases}$ is a Dirichlet character mod M satisfying $\chi_M(1+kN) = 1$ if $(1+kN, M) = 1$.
   — Characters mod M obtained this way with $N < M$ are called *imprimitive*. Other characters are called *primitive*.
   (b) Assume that $\chi_M = \psi_M$ for another character ψ mod N. Show that $\chi = \psi$.
   (c) Show that χ is primitive iff $\bar{\chi}$ is.
   (*d) Let $Q_\chi = \{q \geq 1 \mid \forall n \in \mathbb{Z}\colon\ (n \equiv 1\,(q) \wedge (n,N) = 1) \Rightarrow \chi(n) = 1\}$. Show that if $q_1, q_2 \in Q_\chi$ then their gcd $(q_1, q_2) \in Q_\chi$ as well.
   Hint: Show first that if $q \in Q_\chi$ then $(q,N) \in Q_\chi$.
   (e) Show that there is a primitive character ψ mod $q_\chi = \min Q_\chi$ such that $\chi = \psi_N$.

DEFINITION. Call q_χ the *conductor* of χ.

The Gauss Sum

3. (Fourier analysis on $\mathbb{Z}/N\mathbb{Z}$) Write $L^2(\mathbb{Z}/N\mathbb{Z})$ for the space of $\mathbb{C}$-valued functions on $\mathbb{Z}/N\mathbb{Z}$, equipped with the norm
$$\|f\|_{L^2}^2 = \sum_{a\,(N)} |f(a)|^2 .$$
   For $f \in L^2(\mathbb{Z}/N\mathbb{Z})$ and $k \in \mathbb{Z}/N\mathbb{Z}$ set $\hat{f}(k) = \sum_{a(N)} f(a) e(-\frac{ak}{N})$.
   (a) For $a \in \mathbb{Z}/N\mathbb{Z}$ show that $\sum_{k \in \mathbb{Z}/N\mathbb{Z}} e(\frac{ak}{N}) = N\delta_{a,0}$ (Kronecker delta).
   (b) (Parseval formula) Show that $\left\|\hat{f}\right\|_{L^2} = \sqrt{N}\,\|f\|_{L^2}$.
   (c) (Fourier inversion) Show that $f(a) = \frac{1}{N}\sum_{k\,(N)} \hat{f}(k) e(\frac{ka}{N})$.

4. (The Gauss sum) Let χ be a primitive Dirichlet character mod N. For $b \in \mathbb{Z}/N\mathbb{Z}$ we define the *Gauss sum*

$$G(\chi,b) = \sum_{a\,(N)} \chi(a)e(\frac{ab}{N}).$$

In particular, write $G(\chi) = G(\chi,1)$.

(a) Let $(c,N) = 1$. Show that $G(\chi,bc) = G(\chi,b)\bar{\chi}(c)$. Conclude that if $(b,N) = 1$ we have $G(\chi,b) = G(\chi)\cdot\bar{\chi}(b)$.

(b) Let $(b,N) > 1$. Show that $G(\chi,b) = 0$.

(c) Evaluate $|G(\chi)|$.
Hint: Use Parseval's identity.

(d) Show that $\chi(a) = \frac{G(\chi)}{N}\sum_{b\,(N)}\bar{\chi}(b)e(-\frac{ab}{N})$.

(e) Show that $G(\bar{\chi}) = \chi(-1)\overline{G(\chi)}$.

5. (Twisted Poisson summation) Fix a primitive Dirichlet character χ of conductor q.

(a) For $f \in \mathcal{S}(\mathbb{R}^n)$ set $(D_a f)(x) = f(ax)$, and $(M_p f)(x) = f(x)e(-px)$. Show that $\widehat{D_a f}(k) = \frac{1}{a^n}\hat{f}(\frac{k}{a})$ and $\widehat{M_p f}(k) = \hat{f}(k+p)$.

(b) Prove the *twisted Poisson summation formula*: for $f \in \mathcal{S}(\mathbb{R})$ one has

$$\sum_{n\in\mathbb{Z}} f(n)\chi(n) = \frac{G(\chi)}{q}\sum_{k\in\mathbb{Z}}\hat{f}(\frac{k}{q})\bar{\chi}(k).$$

Hint: use the formula from 4(d).

Dirichlet L-functions

Fix a primitive Dirichlet character χ of conductor q. We assume that $q > 1$, so in particular $\chi(0) = 0$.

6. (cf PS 2, problem 8) For $\varphi \in \mathcal{S}(\mathbb{R})$ and $r > 0$ set $\varphi_\chi(r) = \sum_{n\in\mathbb{Z}}\varphi(rn)\chi(n)$ and $Z(\chi;\varphi;s) = \int_0^\infty \varphi_\chi(q^{-1/2}r)r^s\frac{dr}{r}$.

(a) Show that the sum converges absolutely.

(b) Show that as $r\to\infty$, $\left|\varphi_\chi(r)\right|$ decays faster than any polynomial and that as $r \to 0$, $\left|\varphi_\chi(r)\right| = O(r^{-1})$. Conclude that $Z(\chi;\varphi;s)$ converges absolutely in $\Re(s) > 1$ and defines a holomorphic function there.

(c) Show that for $\Re(s) > 1$,

$$Z(\chi;\varphi;s) = \int_1^\infty \varphi_\chi\left(\frac{r}{\sqrt{q}}\right)r^s\frac{dr}{r} + \frac{G(\chi)}{\sqrt{q}}\int_1^\infty \hat{\varphi}_{\bar{\chi}}\left(\frac{r}{\sqrt{q}}\right)r^{1-s}\frac{dr}{r}$$

(d) Since $\varphi \in \mathcal{S}(\mathbb{R})$ we also have $\hat{\varphi} \in \mathcal{S}(\mathbb{R})$ and $\hat{\hat{\varphi}}(x) = \varphi(-x)$. Conclude that $Z(\chi;\varphi;s)$ extends to an entire function of s which satisfies the *functional equation*

$$Z(\chi;\varphi;s) = \frac{G(\chi)}{\sqrt{q}}Z(\bar{\chi};\hat{\varphi};1-s).$$

(e) Assume that φ has the same parity as χ, that is that $\varphi(-x) = \chi(-1)\varphi(x)$, and show that for $\Re(s) > 1$ we have

$$Z(\chi;\varphi;s) = 2q^{s/2}\left(\int_0^\infty \varphi(r)r^s\frac{dr}{r}\right)L(s;\chi)\,.$$

(f) For $\varphi \in C_c^\infty(\mathbb{R}^\times)$, $\left(\int_0^\infty \varphi(r)r^{ns}\frac{dr}{r}\right)$ extends to an entire function; conclude that $E(\Lambda;s)$ extends to a meromorphic function of s.

(g) Set $a \in \{0,1\}$ so that $\chi(-1) = (-1)^a$. For $\varphi(x) = x^a\exp\left\{-\pi x^2\right\}$ show that $\hat{\varphi}(x) = (-i)^a\varphi(x)$ and that

$$\int_0^\infty \varphi(r)r^s\frac{dr}{r} = 2\Gamma_{\mathbb{R}}(s+a)$$

where $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$. Conclude that $L(s;\chi)$ extends to an entire function, and that the *completed L-function* $\Lambda(s;\chi) = q_\chi^{s/2}\Gamma_{\mathbb{R}}(s+a)L(s;\chi)$ satisfies the functional equation

$$\Lambda(s;\chi) = \varepsilon(\chi)\Lambda(1-s;\bar{\chi})\,,$$

where $\varepsilon(\chi) = q^{-1/2}G(\chi)(-i)^a$. Show that $|\varepsilon(\chi)| = 1$.

Boundedness in vertical strips

OPT (The Phragmén-Lindelöf Theorem) Let $f(s)$ be continuous in the strip $S = \{\sigma + it \mid \sigma \in [a,b]\}$ and holomorphic in the interior of the strip.

(a) (Simple version) Assume that $|f(s)| \leq M$ on ∂S and that f is bounded on S. Then $|f(s)| \leq M$ on S.
Hint: Apply the maximum principle to $f(s)\cdot e^{\varepsilon(s-s_0)^2}$ in an appropriate domain.

— (Functions of finite order) For the rest of the problem we assume that for some $C,A \geq 0$ and all $s \in S$ we have the a-priori bound$|f(s)| \leq Ce^{|s|^A}$.

(b) Show $|f(s)| \leq M$ on ∂S implies $|f(s)| \leq M$ on S.

(c) (Interpolation) Assume that $|f(a+it)| \leq M_a$ and that $|f(b+it)| \leq M_b$. Let $\tau(\sigma)$ be the linear function such that $\tau(a) = 0$, $\tau(b) = 1$. Show that for $a \leq \sigma \leq b$ we have

$$|f(\sigma+it)| \leq M_a^{1-\tau}M_b^\tau\,.$$

Hint: Find a function g such that $|g(s)|$ is precisely the RHS.

(d) (Polynomial growth on the boundary) Assume that $|f(a+it)| \leq M(1+|t|)^\alpha$ and that $|f(b+it)| \leq M(1+|t|)^\beta$. Show that for some $M' \geq 0$ and all $a \leq \sigma \leq b$ we have

$$|f(\sigma+it)| \leq M'(1+|t|)^{(1-\tau(\sigma))\alpha+\tau(\sigma)\beta}\,.$$

(e) Assume now that f is meromorphic in the strip, with only finitely many poles, and that the assumptions of (d) hold away from fixed neighbourhoods of the poles. Show that, away from these neighbourhoods, the conclusion holds up to a loss in the constant.

8. (Boundedness in vertical strips) Let $D(s) = \sum_{n\geq 1} a_n n^{-s}$ be a Dirichlet series which converges absolutely in $\Re(s) > 1$, and write $\bar{D}(s) = \sum_{n\geq 1} \overline{a_n} n^{-s}$. Assume that we have a Γ-factor

$$\gamma(s) = \prod_{j=1}^{d} \Gamma_{\mathbb{R}}(s+\kappa_j)$$

with $\Re(\kappa_j) > -1$ and a conductor q so that $\Lambda(s) = q^{s/2}\gamma(s)D(s)$ is a function of finite order with finitely many poles in the critical strip and satisfies the fuctional equation

$$\Lambda(s) = \varepsilon\bar{\Lambda}(1-s)$$

with $\bar{\Lambda}(s) = q^{s/2}\gamma(s)\bar{D}(s)$. Now that the Γ-factor has no zeroes and by the assumption on the κ_j it has no poles in $\Re(s) \geq 1$.

(a) Use the functional equation to show that $\Lambda(s)$ has no poles in $\Re(s) < 0$.

— Recall that $D(s)$ is bounded on the line $\Re(s) = \sigma$ for any $\sigma > 1$.

(b) Deduce from Stirling's formula that for σ fixed, $\Gamma(s) = \sqrt{2\pi}(it)^{\sigma-\frac{1}{2}}e^{-\frac{\pi}{2}|t|}\left(\frac{|t|}{e}\right)^{it}\left(1+O(\frac{1}{|t|}\right)$.

(c) Show that for $\sigma < 0$ chosen so that we don't hit poles of the Γ-factor we have $|D(\sigma+it)| \leq M(1+|t|)^{\alpha}$.

(d) Show that $D(s)$ extends to a meromorphic function in $\mathbb{C}$ of finite order with poles dividing those of $\Lambda(s)$.

(e) Show that $D(s)$ grows at most polynomially in any vertical strip (away from the its poles).

(f) Conclude that $\Lambda(s)$ decays exponentially in any vertical strip, away from the poles.

5.1. Dirichlet series and modular forms

- Formal Dirichlet series, Dirichlet convolution.
- Formal Euler products (Problem set).

LEMMA 152. *$D(s) = \sum_{n\geq 1} a_n n^{-s}$ converges at one point iff a_n is of at most polynomial growth. In that case:*

- *The series converges absolutely in a right half-plane $\Re(s) > \sigma_a$*
- *The series converges in a right half-plane $\Re(s) > \sigma_c$.*
- *$\sigma_c \leq \sigma_a \leq \sigma_c + 1$.*
- *$D(s)$ defines a holomorphic function in the domain of convergence; this function determines a_n uniquely.*

THEOREM 153. *(Landau) If $a_n \geq 0$ then $D(s)$ has a singularity at σ_a.*

LEMMA 154. *TFAE:*

(1) *$\{a_n\}_{n\geq 0}$ has polynomial growth.*
(2) *$f(z) = \sum_{n\geq 0} a_n e(nz)$ defines a holomorphic function on $\Gamma_\infty\backslash\mathbb{H}$ which is bounded as $y \to \infty$ and grows at most polynomially as $y \to 0$, in both cases uniformly in x.*

Under these hypothesis we have $f(z) - a_0 = O(e^{-2\pi y})$ as $y \to \infty$.

5.2. Hecke theory

For $f(z)$ as above set

$$I_N(s;f) = \int_0^\infty \left[f\left(\frac{iy}{\sqrt{N}}\right) - a_0 \right] y^s \frac{dy}{y}.$$

LEMMA 155. *Assume $a_n \ll n^\nu$ for $\nu \geq 0$. Then $I_N(f;s)$ converges locally uniformly absolutely for $\Re(s) > \nu + 1$.*

PROOF. We have

$$\begin{aligned} |I_N(s;f)| &\leq \sum_{n\geq 1} |a_n| \int_0^\infty e^{-2\pi y/\sqrt{N}} y^\sigma \frac{dy}{y} \\ &\leq \left(\frac{\sqrt{N}}{2\pi}\right)^\sigma \sum_{n\geq 1} |a_n| n^{-\sigma} \int_0^\infty e^{-y} y^\sigma \frac{dy}{y} \\ &= \left(\frac{\sqrt{N}}{2\pi}\right)^\sigma \Gamma(\sigma) \sum_{n\geq 1} |a_n| n^{-\sigma} < \infty. \end{aligned}$$

□

It follows that in the half-plane $\Re(s) > \nu + 1$ we have

$$I_N(f;s) = N^{s/2}(2\pi)^{-s}\Gamma(s)\cdot D(s).$$

THEOREM 156. *(Hecke) Let $f = \sum_{n\geq 0} a_n e(nz)$, $g = \sum_{n\geq 0} b_n e(nz)$. Then TFAE:*

(1) $g(z) = \left(\sqrt{N}z\right)^{-k} f(-1/Nz) = f|_k \omega_N$.

(2) *$I_N(s;f)$ and $I_N(s;g)$ continue meromorphically to $\mathbb{C}$with $I_N(s;f)+\frac{a_0}{s}+\frac{i^k b_0}{k-s}$ entire. and satisfy the* functional equation

$$I_N(s;f)=i^k I_N(k-s;g)\,.$$

PROOF. Set

$$I_N^{(1)}(s;f)=\int_1^\infty\left[f\left(\frac{iy}{\sqrt{N}}\right)-a_0\right]y^s\frac{dy}{y}$$

and note that this defines an entire function. In the domain of absolute convergence we then have

$$\begin{aligned}
I_N(s;f) &= \int_1^\infty\left[f\left(\frac{iy}{\sqrt{N}}\right)-a_0\right]y^s\frac{dy}{y}+\int_0^1\left[f\left(\frac{iy}{\sqrt{N}}\right)-a_0\right]y^s\frac{dy}{y}\\
&= I_N^{(1)}(s;f)-\frac{a_0}{s}+i^k\int_0^1 y^{-k}g\left(\frac{i}{y\sqrt{N}}\right)y^s\frac{dy}{y}\\
&= I_N^{(1)}(s;f)-\frac{a_0}{s}+i^k\int_1^\infty g\left(\frac{iy}{\sqrt{N}}\right)y^{k-s}\frac{dy}{y}\\
&= I_N^{(1)}(s;f)-\frac{a_0}{s}-\frac{i^k b_0}{k-s}+i^k I_N^{(1)}(k-s;g)\,.
\end{aligned}$$

□

This gives AC. For FE apply the same formula to $I_N(s;g)$.

For the converse, let $\Lambda(s;f)=N^{s/2}(2\pi)^{-s}\Gamma(s)\cdot D_f(s)$, $\Lambda(s;f)=N^{s/2}(2\pi)^{-s}\Gamma(s)\cdot D_g(s)$ and assume that these satisfy AC, FE. In PS6 we show that AC+FE make $D_f(s)$ be polynomially BVS and hence $\Lambda(s;f)$ decays exponentially in vertical strips. It follows the integral

$$\frac{1}{2\pi i}\int_{\Re(s)=\alpha}\Lambda(s;f)(\sqrt{N}y)^{-s}ds$$

converges absolutely if the line does not hit a pole, and for $\Re(\alpha)$ large absolute convergence of the Dirichlet series and Fourier expansion gives

$$f(iy)=a_0+\frac{1}{2\pi i}\int_{\Re(s)=\alpha}\Lambda(s;f)(\sqrt{N}y)^{-s}ds$$

since

$$\frac{1}{2\pi i}\int_{\Re(s)=\alpha}(2\pi ny)^{-s}\Gamma(s)ds=e^{-2\pi ny}\,.$$

The exponential decay in vertical strips allows us to shift the contour to $\Re(s)=k-\alpha$ and conclude:

$$\begin{aligned}
f(iy) &= a_0+\frac{1}{2\pi i}\int_{\Re(s)=k-\alpha}I_N(s;f)(\sqrt{N}y)^{-s}ds-a_0+i^k b_0(\sqrt{N}y)^{-k}\\
&= i^k b_0(\sqrt{N}y)^{-k}+(\sqrt{N}y)^{-k}\frac{1}{2\pi i}\int_{\Re(s)=\alpha}i^k I_N(s;g)(\sqrt{N}y)^{-s}ds\\
&= (-i\sqrt{N}y)^{-k}g(-1/iNy)\,.
\end{aligned}$$

Now $g(z), f|_k\omega_N$ are holomorphic so equality on the imaginary axis implies equality everywhere.

COROLLARY 157. *k even. Then $f(z) \in \mathcal{H}_k(1)$ iff $I(s;f) = (2\pi)^{-s}\Gamma(s)D(s)$ has AC with poles $-\frac{a_0}{s} - \frac{(-1)^{k/2}a_0}{k-s}$, and FE*

$$I(s;f) = (-1)^{k/2}I(k-s;f)\,.$$

5.3. Weil's converse theorem

5.3.1. Twists. Let $E : y^2 = x^3 + ax + b$ be an elliptic curve with integral coefficients. Recall the definition $a_p(E) = p + 1 - \#E(\mathbb{F}_p)$ for primes of good reduction and the Hasse-Weil L-function

$$L(s;E) = \prod_p \left(1 - a_p p^{-s} + 1_{N_E}(p)p^{-2s}\right)^{-1}$$

where $1_{N_E}(p) = \begin{cases} p & (p,N_E) = 1 \\ 0 & p \mid N_E \end{cases}$. For $p \| N_E$ we have $a_p(E) = \pm 1$ and for $p^2 \| N_E$ we have $a_p(E) = 0$.

THEOREM 158. *(Modularity Theorem; W,TW,BCDT) There is a Hecke newform $f \in \mathcal{S}_2(N_E)$ such that $L(s;E) = L(s;f)$. In particular, $L(s;E)$ is* nice.

For d squarefree and not divisible by $2,3$ consider the elliptic curve

$$E_d : dy^2 = x^3 + ax + b$$

Note that E, E_d are isomorphic over $\mathbb{Q}(\sqrt{d})$. What is $a_p(E_d)$? If d is a square mod p, then E_d and E are isomorphic over $\mathbb{F}_p$. If d is not a square, then for every finite $x \in \mathbb{F}_p$ such that $y \neq 0$, exactly one of $(x^3 + ax + b), d(x^3 + ax + b)$ is a square. It follows that $\#E(\mathbb{F}_p) + \#E_d(\mathbb{F}_p) = 2(p+1)$, that is that $a_p(E_d) = -a_p(E)$. We have concluded that $a_p(E) = \chi_d(p)a_p(E)$ where $\chi_d = \left(\frac{d}{\cdot}\right)$ is the quadratic character mod d. In other words, if

$$L(s;E) = \sum_{n\geq 1} a_n(E)n^{-s}$$

then

$$L(s;E_d) = \sum_{n\geq 1} a_n(E)\chi_d(n)n^{-s}\,.z$$

5.3.2. Direct thm with twists. Fix a primitive Dirichlet char ψ, conductor $m = m_\psi$.
Given the sequence $\{a_n\}$ we set

$$\begin{aligned} f_\psi(z) &= \sum_{n\geq 1} a_n \psi(n) e(nz) \\ D_\psi(s) &= \sum_{n\geq 1} a_n \psi(n) n^{-s} \end{aligned}$$

and set

$$I_N(s; f \times \psi) \stackrel{\text{def}}{=} I_{Nm^2}(s; f_\psi) = (2\pi/m\sqrt{N})^{-s}\Gamma(s)D_\psi(s)\,.$$

LEMMA 159. *Let $(m,N) = 1$. Then TFAE*

(1) $C_\psi g_{\bar{\psi}} = f_\psi|_k \omega_{Nm^2}$.
(2) $I_N(s; f \times \psi)$ *and* $I_N(s; g \times \bar{\psi})$ *have AC (no poles)* and the FE , *decays exponentially*

$$I_N(s; f \times \psi) = i^k C_\psi I_N(k-s; g \times \bar{\psi})\,.$$

PROOF. Apply THM to $f_\psi, C_\psi g_{\bar\psi}$. $\square$

LEMMA 160. *$f \in \mathcal{H}_k(N,\chi)$. Then $f_\psi \in \mathcal{H}_k(M, \chi\psi^2)$ where $M = \operatorname{lcm}\left\{N, m_\psi^2, m_\psi m_\chi\right\}$.*

PROOF. Write $m = m_\psi$. Then $f_\psi = G(\bar\psi)^{-1}\sum_{r(m)} \bar\psi(r) f|_k \begin{pmatrix} 1 & r/m \\ & 1 \end{pmatrix}$. Write $u(x) = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}$. Then $u(\frac{b}{m})^{-1}\Gamma_0(N)u(\frac{b}{m}) \supset \Gamma(Nm^2)$ so $f_\psi \in \mathcal{H}_k(\Gamma(Nm^2))$. Now for $\gamma = \begin{pmatrix} a & b \\ Mc & d \end{pmatrix}$ note that

$$\begin{aligned}
\gamma' &= u(\frac{r}{m})\gamma u(-\frac{d^2 r}{m}) \\
&= \begin{pmatrix} 1 & r/m \\ & 1 \end{pmatrix}\begin{pmatrix} a & b \\ Mc & d \end{pmatrix}\begin{pmatrix} 1 & -d^2 r/m \\ & 1 \end{pmatrix} \\
&= \begin{pmatrix} a + rMc/m & b + dr/m \\ Mc & d \end{pmatrix}\begin{pmatrix} 1 & -d^2 r/m \\ & 1 \end{pmatrix} \\
&= \begin{pmatrix} a + rMc/m & b + dr/m(1-ad) - r^2 d^2 Mc/m^2 \\ Mc & d - d^2 rcM/m \end{pmatrix} \\
&= \begin{pmatrix} a + rcM/m & b - cdrM/m - r^2 d^2 cM/m^2 \\ Mc & d - d^2 rcM/m \end{pmatrix} \in \Gamma_0(M)
\end{aligned}$$

And $d' \equiv d\,(m_\chi)$ so $\chi(\gamma') = \chi(\gamma)$. Thus

$$f|_k u(\frac{r}{m})\gamma = \chi(d) f|_k u(\frac{d^2 r}{m}).$$

Summing over r we find

$$f_\psi|_k\gamma = \psi(d^2)\chi(d) f_\psi.$$

If f is a cusp form so are $f|_k u(r/m)$ and hence f_ψ. $\square$

PROPOSITION 161. *$f \in \mathcal{H}_k(N,\chi)$, ψ primitive of conductor m, $(m,N) = 1$. Then $f_\psi|_k\omega_{Nm^2} = C_\psi g_{\bar\psi}$ where $g = f|_k\omega_N$ and*

$$C_\psi = \chi(m)\psi(-N)\frac{G(\psi)}{G(\bar\psi)} = \chi(m)\psi(-N)\frac{G(\psi)^2}{m}. \tag{5.3.1}$$

PROOF. If $nm - Nrs = 1$ then $u(r/m)\omega_{Nm^2} = m\omega_N \begin{pmatrix} m & -s \\ -rN & n \end{pmatrix} u(s/m)$. $\square$

COROLLARY 162. *Let $f \in \mathcal{S}_k(N,\chi)$, ψ primitive of conductor m, $(m,N) = 1$. Then $I_N(s; f\times\psi)$ is entire and satisfies the FE*

$$I_N(s; f\times\psi) = i^k C_\psi I_N(k-s; f|_k\omega_N \times \bar\psi),$$

with C_ψ as in (5.3.1).

5.3.3. Converse theorem. If $(m,rN)=1$ let n,s such that $mn-Nrs=1$ set $\gamma_{m,r}=\begin{pmatrix} m & -r \\ -sN & n \end{pmatrix}\in$ $\Gamma_0(N)$. We have seen:

$$u(r/m)\omega_{Nm^2}=m\omega_N\gamma_{m,r}u(s/m)\,.$$

LEMMA 163. *If f,g satisfy AC, FE for all ψ mod m prime then*

$$g|_k\left[\chi(m)-\gamma(m,r)\right]u(r/m)$$

is independent of r.

PROOF. $\sum_{r=1}^m\bar{\psi}(s)f|_k\alpha(s/m)\omega_{Nm^2}=\chi(m)\psi(-N)\sum_{s=1}^m\psi(s)g|_ku(s/m)$ .It follows that

$$\sum_{r(m)}\psi(r)g|_k\left[\chi(m)-\gamma(m,r)\right]u(r/m)=0\,.$$

It follows that the function $g|_k\left[\chi(m)-\gamma(m,r)\right]u(r/m)$ is constant. $\square$

LEMMA 164. *Let $(mn,N)=1$, let f,g satisfy the conclusion of Corollary* (162) *for all characters mod m and n. Then for all $\gamma=\begin{pmatrix} m & -s \\ -rN & n \end{pmatrix}$*

PROOF. Let $\gamma'=\begin{pmatrix} m & s \\ rN & n \end{pmatrix}$.

$$g|_k\left[\chi(m)-\gamma\right]u(r/m)=g|_k\left[\chi(m)-\gamma'\right]u(-r/m)$$

that is

$$g|_k\left[\chi(m)-\gamma\right]=g|_k\left[\chi(m)-\gamma'\right]u(-2r/m)$$

Also, $\gamma^{-1}=\begin{pmatrix} n & s \\ rN & m \end{pmatrix}$, $\gamma'^{-1}=\begin{pmatrix} n & -s \\ -rN & m \end{pmatrix}$ satisfy

$$g|_k\left[\chi(n)-\gamma'^{-1}\right]=g|_k\left[\chi(n)-\gamma^{-1}\right]u(-2r/m)\,.$$

Since $\chi(nm)=1$, $\chi(n)-\gamma'^{-1}=-\chi(n)\left(\chi(m)-\gamma'\right)\gamma'^{-1}$ so $\left[\chi(n)-\gamma^{-1}\right]u(-2r/m)=-\chi(n)\left(\chi(m)-\gamma\right)\gamma^{-1}u(-2$
It follows that

$$g|_k\left[\chi(m)-\gamma'\right]=g|_k\left(\chi(m)-\gamma\right)\gamma^{-1}u(-2r/m)\gamma'$$

$\square$

5.4. The Euler product

5.5. The Rankin-Selberg L-function

5.6. Modularity

The Modularity Theorem
Artin's Conjecture

Langlands Conjectures
Serre's conjecture

CHAPTER 6

Analytic bounds

6.1. Fourier coefficients

6.2. The circle problem

CHAPTER 7

Topics

7.1. Hilbert modular forms

7.2. Siegel modular forms

Bibliography