FOURIER SERIES AND THE POISSON SUMMATION FORMULA (NOTES FOR MATH 613)

LIOR SILBERMAN

NOTATION

Write S^1 for the group $\{z \in \mathbb{C}^\times \mid |z| = 1\}$. For $z \in \mathbb{C}$ write $e(z) \stackrel{\text{def}}{=} e^{2\pi i z}$. All group homomorphisms are assumed to be continuous.

For a topological space X write $C(X)$ for the space of $\mathbb{C}$-valued continuous functions on X, $C_c(X)$ for the subspace of functions of compact support. If μ is a Radon measure on X and $1 \leq p \leq \infty$ write $L^p(\mu)$ for the usual space of p-integrable functions. We sometimes write $L^p(X)$ when the measure is clear (and note that if $L^p(f\mu) = L^p(\mu)$ if f is bounded)

When X is compact, $C(X)$ is complete in the L^∞ norm and (Stone-Weierstrass) a subalgebra $\mathcal{A} \subset C(X)$ is dense if it separates points, does not have a common zero, and is closed under conjugation.

On a manifold X write $C^j(X)$ for the space functions differentiable j times with continuous derivatives of order j, $C^\infty(X) = \cap_j C^j(X)$, and $C_c^\infty(X) = C^\infty(X) \cap C_c(X)$.

On $\mathbb{R}^n$ say f is of *rapid decay* if $f(x)\,(1 + \|x\|)^N$ is bounded for all N. Say $f \in C^\infty(\mathbb{R}^n)$ is of *Schwartz class* if f and all its derivatives are of rapid decay.

1. FOURIER SERIES AND FOURIER INVERSION ON $\mathbb{R}^n/\Lambda$

Let V be an inner product space, fix a lattice $\Lambda < V$, and write $\mathbb{T}$ for the torus V/Λ. Let Λ^* be the dual lattice.

Definition 1. $L^2(\mathbb{T})$ and $L^2(\Lambda^*)$ will denote the spaces with respect to the Haar probability measure and counting measure, respectively.

Problem 2. (Functional analysis)

1. Show that $C(\mathbb{T})$ is dense in $L^2(\mathbb{T})$.
2. Show that $C_c(\Lambda^*)$ is dense in $L^2(\Lambda^*)$.

Problem 3. (Trigonometric polynomials)

1. Show that $k \mapsto (x \mapsto e(kx))$ is an injective group homomorphism $\Lambda^* \hookrightarrow \mathrm{Hom}(\mathbb{T}, S^1)$.
2. Show that the characters $e(kx)$ are linearly independent in $C(\mathbb{T})$.
   Hint: Evaluate a linear combination $\sum a_k e(kx) = 0$ of shortest length at two different values of x.
3. Let $\mathcal{P}$ be the algebra of continuous functions on $\mathbb{T}$ generated by the $e(kx)$. Show that $\mathcal{P}$ is simply the linear span of these characters.
4. Let $x \in \mathbb{T}$ be non-zero. Show that there exists $k \in \Lambda'$ such that $e(kx) \neq 1$.
   Hint: $\Lambda^{**} = \Lambda$.

(5) Show that $\mathcal{P}$ separates the points of $\mathbb{T}$ and contains 1. By the Stone-Weierstrass Theorem it follows that $\mathcal{P}$ is dense in $C(\mathbb{T})$.

Problem 4. (Orthogonality of characters)

(1) For $k \in \Lambda^*$ show that $\frac{1}{\mathrm{vol}(\mathbb{T})}\int_{\mathbb{T}} e(kx)dx = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$.

(2) Conclude that for $k, l \in \Lambda^*$ one has $\frac{1}{\mathrm{vol}(\mathbb{T})}\int_{\mathbb{T}} e(kx)\overline{e(lx)}dx = \delta_{kl}$.

Definition 5. For $g \in C_c(\Lambda^*)$ set $\check{g}(x) = \sum_{k\in\Lambda^*} e(kx)$.

Problem 6. (The inverse map) We show that $g \mapsto \check{g}$ extends to an isometric isomorphism $L^2(\Lambda^*) \to L^2(\mathbb{T})$.

(1) (Parseval's identity) Show that $\|\check{g}\|_{L^2(\mathbb{T})} = \|g\|_{L^2(\Lambda^*)}$, that is that $\frac{1}{\mathrm{vol}(\mathbb{T})}\int_{\mathbb{T}} |\check{g}(x)|^2 dx = \sum_{k\in\Lambda^*} |g(k)|^2$.

(2) Since $C_c(\Lambda^*)$ is dense in $L^2(\Lambda^*)$, conclude that $g \to \check{g}$ extends to an isometric embedding $L^2(\Lambda^*) \to L^2(\mathbb{T})$, and show that the image is a closed subspace.

(3) Let $f \in L^2(\mathbb{T})$ be of norm one and orthogonal to the image of this map. Approximating f by a trigonometric polynomial show that $(f, f) = 0$ and derive a contradiction.

Definition 7. For $f \in L^2(\mathbb{T})$ and $k \in \Lambda^*$ set $\hat{f}(k) = \frac{1}{\mathrm{vol}(\mathbb{T})}\int_{\mathbb{T}} f(x)e(-kx)dx$.

Problem 8. (The direct map)

(1) Show that $\left|\hat{f}(k)\right| \leq \|f\|_{L^2(\mathbb{T})}$. Conclude that $\left|\hat{f}(k)\right| \leq \|f\|_{L^\infty(\mathbb{T})}$ also.

(2) For $g \in C_c(\Lambda^*)$ show that $\hat{\check{g}}(k) = g(k)$. Show that the same holds for $g \in L^2(\Lambda^*)$.

(3) Conclude that the map $f \mapsto \hat{f}$ takes values in $L^2(\Lambda^*)$ and is the inverse to the map $g \mapsto \check{g}$.

Problem 9. (Smooth functions)

(1) Integrating by parts, show that for $k \neq 0$ and $f \in C^{2j}(\mathbb{T})$ we have $\left|\hat{f}(k)\right| \leq \frac{1}{|2\pi k|^j}\left\|\triangle^j f\right\|_{L^\infty(\mathbb{T})}$.

(2) Assume now that $f \in C^\infty(\mathbb{T})$. Show that $F^{(\alpha)}(x) = \sum_{k\in\Lambda^*}(2\pi ik)^\alpha \hat{f}(k)e(kx)$ converges uniformly for all multi-indices α.

(3) Integrating term-by-term show that $F^{(\alpha)}$ is the αth derivative of $F^{(0)}$.

(4) Show that $F^{(0)} = f$ pointwise.

2. The Poisson Summation Formula

Definition 10. For $f \in L^1(V)$ and $k \in V^*$ set $\hat{f}(k) = \int_V f(x)e(-kx)dx$ and call this the *Fourier transform* of f.

Problem 11. (The Fourier transform) Let $f \in L^1(\mathbb{R}^n)$

(1) Show that $\left\|\hat{f}\right\|_{L^\infty(V^*)} \leq \|f\|_{L^1(V)}$.

(2) Show that $\hat{f} \in C(V)$.
Hint: The bounded convergence theorem.

(3) On $V = \mathbb{R}$ let $f = \exp(-|x|)$. Show that $\hat{f}(k) = \frac{2}{1+4\pi^2k^2}$.

(4) Let $\Re(\alpha) > 0$ and let $f(x) = \exp\left\{-\pi\alpha x^2\right\}$. Show that $\hat{f}(k) = \sqrt{\frac{1}{\alpha}}\exp\left\{-\frac{\pi}{\alpha}k^2\right\}$ where we take the branch of the square root with a cut at $(-\infty, 0]$.
Hint: Shift contours to reduce the problem to the known formula $\int_{\mathbb{R}} \exp\left(-\alpha x^2\right) dx = \sqrt{\frac{\pi}{\alpha}}$.

(5) Let $Q \in M_n(\mathbb{R})$ be a positive-definite symmetric matrix, and let $f(x) = \exp\left(-2\pi \langle x | Q | x \rangle\right)$. Show that $\hat{f}(k) = 2^{-n/2} (\det Q)^{-1/2} \exp\left\{-2\pi \left\langle k \left| Q^{-1} \right| k \right\rangle\right\}$.

Finally, let $f \in C^\infty(\mathbb{R}^n)$ and its derivatives decay polynomially, quickly enough that $\Pi_\Lambda f$ converges absolutely to a smooth function.

Problem 12. (The Poisson Summation Formula) Let $f \in L^1(\mathbb{R}^n)$ decay quickly enough that $\Pi_\Lambda f \in \mathbb{C}(V/\Lambda)$.

(1) For $k \in \Lambda^*$ show that $\widehat{\Pi_\Lambda f}(k) = \frac{1}{\operatorname{vol}(\Lambda)} \hat{f}(k)$ where the first hat is the Fourier transform on $\mathbb{T}$ and the second is the one on V.

(2) Show that $\Pi_\Lambda f(x) = \frac{1}{\operatorname{vol}(\Lambda)} \sum_{\Lambda^*} \hat{f}(k) e(kx)$. Conclude that:

$$\sum_{v \in \Lambda} f(v) = \frac{1}{\operatorname{vol}(\Lambda)} \sum_{k \in \Lambda^*} f(k)\,.$$

3. The Fourier transform and Fourier inversion on $\mathbb{R}^n$

Problem 13. (The Schwartz class) Let $f \in \mathcal{S}(V)$, $\Lambda < V$ a fixed lattice.

(1) Differentiating under the integral sign show that $\hat{f}(k)$ is smooth.

(2) Integrating by parts show that then $\hat{f}$ is of rapid decay.

(3) Combining the two calculations show that $\hat{f} \in \mathcal{S}(V)$.

(4) Applying the PSF to f with the lattice $r\Lambda$ and taking $r \to \infty$ show that

$$f(0) = \int_{V^*} \hat{f}(k) dk\,.$$

(5) Let $g(x) = f(x+y)$. Show that $\hat{g}(k) = \hat{f}(k) e(ky)$ and conclude that

$$f(x) = \int_{V^*} \hat{f}(k) e(kx) dk\,.$$

(6) Use the same methods to establish *Parseval's identity*

$$\|f\|_{L^2(V)} = \left\|\hat{f}\right\|_{L^2(V^*)}\,.$$

(7) Conclude that the Fourier transform extends to a bijective isometry $\mathcal{F}\colon L^2(V) \to L^2(V^*)$.