Math 121: Problem set 3 (due 27/1/12)

Practice problems (not for submission!)

Section 6.1-6.3: all problems, especially those marked "challenging". Ignore problems for computer-assisted exploration.

Estimation

1. Simplifying integrals.
   (a) Show that $\int_1^2 \sqrt{x^2-1}\,\mathrm{d}x \le \frac{3}{2}$.
   Hint: $\sqrt{x^2-1} \le \sqrt{x^2}$.
   (b) Evaluate $\lim_{T\to\infty} \frac{1}{T}\int_T^{T+1} \sqrt{x^2-1}\,\mathrm{d}x$.

2. Estimating $\log(2)$ and π.
   (a) Show that $\int_0^1 \frac{1}{1+t^2}\,\mathrm{d}t = \frac{\pi}{4}$.
   Hint: What is $\frac{d}{dx}(\arctan x)$?
   (b) Show that $1-t^2 \le \frac{1}{1+t^2} \le 1-t^2+t^4$ and conclude that $\frac{8}{3} \le \pi \le \frac{52}{15}$.
   (c) Show that $\int_0^1 \frac{(x-x^2)^2}{1+x^2}\,\mathrm{d}x = \log(2) - \frac{2}{3}$
   Hint: Write $(x-x^2)^2$ in the form $2x+(1+x^2)P(x)$ where $P(x)$ is a polynomial.
   (d) Show that $\frac{2}{3} \le \log(2) \le \frac{2}{3} + \frac{1}{30}$.
   Hint: For the upper bound, note that $\frac{1}{1+x^2} \le 1$ for all x.
   SUPP Show that $\int_0^1 \frac{(x-x^2)^4}{1+x^2}\,\mathrm{d}x = \frac{22}{7} - \pi$.
   SUPP Show that $\frac{1979}{630} \le \pi \le \frac{22}{7}$.

3. Let f,g be continuous on the interval $[a,b]$.
   SUPP Show that $\int_a^b (f(x))^2\,\mathrm{d}x = 0$ implies that $f(x)=0$ for all x.
   Hint: Assuming f is non-zero somewhere show that it is non-zero on an entire subinterval, and construct a non-zero lower Riemann sum for f^2 on $[a,b]$.
   (b) Assuming that f is not identically zero, find the point t_0 where the function $G(t)$ below achieves its global minimum.
   $$G(t) = \int_a^b (tf(x)+g(x))^2\,\mathrm{d}x.$$
   (c) Show that $G(t_0) \ge 0$ and deduce the *Cauchy-Schwartz inequality*
   $$\left(\int_a^b f(x)g(x)\,\mathrm{d}x\right)^2 \le \left[\int_a^b (f(x))^2\,\mathrm{d}x\right]\left[\int_a^b (g(x))^2\,\mathrm{d}x\right].$$
   – What about the case where f is identically zero?

Techniques of integration

4. Let $I_n = \int x^{2n}\cos x\,\mathrm{d}x$, $J_n = \int \sin^n x\,\mathrm{d}x$.
   (a) Obtain a reduction formula for I_n.
   Hint: Textbook page 335.
   (b) Obtain a reduction formula for J_n.
   Hint: Textbook page 336.
   (c) Use your formula to calculate $\int_{-\pi/2}^{\pi/2} x^{2n}\cos x\,\mathrm{d}x$ for $n=3$.

Supplementary problem – the substitution rule for discontinuous functions

A. Let $[a,b]$ be an interval, and let $g\colon [a,b] \to \mathbb{R}$ be continuously differentiable with positive derivative. Let $c = g(a)$ and $d = g(b)$.

(a) Show that g is surjective ("onto") the interval $[c,d]$: that for $u \in [c,d]$ there is $x \in [a,b]$ with $g(x) = u$.

(b) Show that g is injective ("one-to-one"): the x in part (a) is unique. We write $g^{-1}(u)$ for the unique x solving $g(x) = u$.

(c) Let $P : a = x_0 < \cdots < x_n = b$ be a partition of $[a,b]$. Show that setting $u_i = g(x_i)$ gives a partition of $[c,d]$, to be denoted $g(P)$.

(**d) Let f be bounded on $[c,d]$ and let $\varepsilon > 0$. Show that if the mesh $\delta(P)$ is small enough then $|U(f;g(P)) - U((f \circ g)g';P)| \leq \varepsilon$ and $|L(f;g(P)) - L((f \circ g)g';P)| \leq \varepsilon$.

(e) Suppose that $(f \circ g)g'$ is integrable on $[a,b]$, or that f is integrable on $[c,d]$. Show that the other function is integrable as well and that $\int_a^b f(g(x))g'(x)\,\mathrm{d}x = \int_c^d f(u)\,\mathrm{d}u$.

RMK Note that f was not assumed continuous.