Math 121: Problem set 5 (due 10/2/12)

Practice problems (not for submission!)

Section 6.5

Improper integrals

1. In both part (a) and part (b) use the change-of-variable $x = -y$ to show that:
   (a) Suppose that $f(x) = f(-x)$ for all x (f is "even"). Show that $\int_{x=-a}^{x=0} f(x)\,dx = \int_{x=0}^{x=a} f(x)\,dx$ and therefore that $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.
   (b) Suppose that $f(x) = -f(-x)$ for all x (f is "odd"). Show that $\int_{-a}^{a} f(x)\,dx = 0$.

SUPP (Why we treat each boundary point separately) Consider the integral $\int_{-1}^{1} \frac{1}{x}\,dx$.
RMK $\frac{1}{x}$ is odd, and the point is to understand why the integral is undefined rather than zero.
(a) Show that $\int_{-1}^{-\varepsilon} \frac{1}{x}\,dx + \int_{\varepsilon}^{1} \frac{dx}{x} = 0$ (we removed a symmetric neigbourhood of the bad point $x = 0$).
(b) Show that $\lim_{\varepsilon\to 0^+} \int_{-1}^{-\varepsilon} \frac{dx}{x} + \int_{\varepsilon^2}^{1} \frac{dx}{x} = \infty$ (we removed the neighbourhood $\left(-\varepsilon, \varepsilon^2\right)$ which is "lop-sided")
(c) Show that $\lim_{\varepsilon\to 0^+} \int_{-1}^{-\varepsilon^2} \frac{dx}{x} + \int_{\varepsilon}^{1} \frac{dx}{x} = -\infty$ (we removed the neighbourhood $\left(-\varepsilon^2, \varepsilon\right)$).

2. Show that the following integrals converge absolutely.
   (a) $\int_5^\infty e^{-x}\,dx$.
   (b) $\int_{10}^\infty \frac{\cos(e^x)e^{-x^2}}{1+x^2-\sin x}\,dx$.

3. In this problem we evaluate the integral $I = \int_0^\infty \frac{\sin x}{x} dx$ by a method commonly used in physics called "regularization" or "adding a convergence factor".
   (a) Show that $\left|\frac{\sin x}{x}\right| \le 1$ for all x.
   Hint: Apply the mean value theorem to $\frac{\sin x - \sin 0}{x-0}$.
   (b) Conclude that $\int_0^1 \frac{\sin x}{x}\,dx$ converges.
   Hint: Absolute convergence.
   (c) Using integration by parts, show that $\int_1^T \frac{\sin x}{x}\,dx = -\left[\frac{\cos x}{x}\right]_{x=1}^{x=T} - \int_1^T \frac{\cos x}{x^2}\,dx$. Taking the limit as $T \to \infty$ show that $\int_1^\infty \frac{\sin x}{x}\,dx$ converges.
   RMK Integration by parts "allows us" to see the cancellation.
   SUPP Show that $\int_0^\infty \left|\frac{\sin x}{x}\right| dx$ diverges.
   Hint: Consider intervals of the form $\left[2\pi k + \frac{\pi}{4}, 2\pi k + \frac{3\pi}{4}\right]$.
   RMK This gives us an example of an integral that converges, but not absolutely.
   (d) For $t > 0$ set $I(t) = \int_0^\infty \frac{\sin x}{x} e^{-tx} dx$. Show that $I(t)$ converges absolutely and that $|I(t)| \le \frac{1}{t}$.
   Hint: $\left|\frac{\sin x}{x} e^{-tx}\right| \le e^{-tx}$.
   (e) Show that $\lim_{t\to\infty} I(t) = 0$.
   (f) Show that $0 \le 1 - e^{-t} \le t$ for all $t \ge 0$.
   Hint: The mean value theorem.
   (g) Show that $\lim_{t\to 0} \int_0^T \frac{\sin x}{x} e^{-tx}\,dx = \int_0^T \frac{\sin x}{x}\,dx$.
   Hint: Subtract the integrals of both sides and combine the idea of (d) and the result of (f).
   SUPP In fact, $\lim_{t\to 0} I(t) = I$.

SUPP ("Differentiation under the integral sign") Show that for $t > 0$ $I(t)$ is differentiable as a function of t and that

$$I'(t) = \int_0^\infty \frac{\sin x}{x}\left(-xe^{-tx}\right)\mathrm{d}x$$

(note that the term in the parentheses is the derivative of e^{-tx} with respect to t).

Hint: This is a harder version of (g): You need to give an upper bound on $\left|\frac{e^{-(t+h)x}-e^{-tx}}{h} - (-xe^{-tx})\right|$ which is good enough to take the limit as $h \to 0$.

(h) Evaluate $J(t) = \int_0^\infty e^{-tx}\sin x\,\mathrm{d}x$ using a double integration by parts.

(i) The supplementary part showed that $I'(t) = -J(t)$. Find a formula for $I(t)$ by integrating your answer from part (h); use part (e) for the constant of integration.

(j) Take the limit $t \to 0$ in your formula to find the value of I.

Supplementary problems

A. (Monotone convergence principle)

(a) Suppose that $f(x)$ is monotone nondecreasing and bounded above on $[a,b)$. Show that $\lim_{x\to b} f(x)$ exists.

Hint: The limit is $\sup\{f(x) \mid a \le x < b\}$.

(b) Suppose that $f(x)$ is monotone nondecreasing on $[a,b)$ but not bounded above. Show that $\lim_{x\to b} f(x) = \infty$ in the extended sense.

(c) Repeat parts (a),(b) for f monotone on the interval $[a,\infty)$, studying $\lim_{x\to\infty} f(x)$.

B. (Convergence is not just about asymptotic rate of decay) For $x \ge 1$ let $f(x) = \begin{cases} x & x \le \lfloor x \rfloor + \frac{1}{\lfloor x \rfloor^3} \\ 0 & \text{otherwise} \end{cases}$ and $g(x) = \begin{cases} x & x \le \lfloor x \rfloor + \frac{1}{\lfloor x \rfloor} \\ 0 & \text{otherwise} \end{cases}$ where $\lfloor x \rfloor$ is the largest integer which is not greater than x (e.g. $\lfloor 5 \rfloor = \lfloor 5.5 \rfloor = 5$).

(a) Show that $f(x) \le g(x)$ for all $x \ge 1$.

(b) Show that $f(x)$ does not decay at all as $x \to \infty$.

(c) Show that $\int_1^\infty f(x)\,\mathrm{d}x$ converges and that $\int_1^\infty g(x)\,\mathrm{d}x$ diverges.

RMK This is a rather pathological example; functions arising in "real life" don't normally behave like this.