Math 121: Problem set 7 (due 6/3/12)

Practice problems (not for submission!)

Section 7.7 - Probability
Section 7.8 - Separable equations only

Differential Equations

1. Find a function $y(x)$ so that $y' = \cos y$ and $y(0) = 0$.
2. There is a function f, defined for $x > 0$, for which $f(x) - \int_0^x \frac{f(t)}{t^2}\,dt$ is constant and such that $f(1) = \frac{1}{e}$.
   (a) Supposing f exists, find it.
   (*b) Show that the f you found actually solves the equation, in that the improper integral converges.

Probability

3. Let X be distributed among $\{0, 1, \ldots, n-1\}$ where $\Pr(X = i)$ is propotional to q^i (here $0 < q < 1$ is a constant.).
   (a) Find the constant C so that $\Pr(X = i) = Cq^i$.
   Hint: The total probability must be 1; now use Problem set 1, problem 2(b).
   (b) Find the expectation of X.
   Hint: PS1, Problem 2(c).
   (c) Show that as $n \to \infty$ the answer of (b) tends to a finite limit. In other words, for n very large X occasionally takes large values, but these occur rarely enough to keep the expectations bounded.
   SUPP Find the variance.
4. For each of the following functions f find a normalizing constant so that $p(x) = cf(x)$ is a probability density function. Now let $m_n(f) = \int_a^b x^n p(x)\,dx$ denote the "nth moment" of p. Next, calculate (2) $\mu = m_1(f)$ (3) $\sigma = \sqrt{m_2(f) - (m_1(f))^2}$ (4) the "moment generating function" $M(t) = \mathbb{E}e^{tX} = \int_a^b e^{tx} p(x)\,dx$ (in particular, find for which values of t this converges).
   (a) ("Gamma distribution") $f(x) = x^{s-1}e^{-x}$ on the interval $0 < x < \infty$ and zero otherwise ($s > 0$ is a fixed parameter).
   Hint: μ, σ^2 are polynomials in s.
   (b) For $s > 1$ find the location of the peak of the Gamma distribution. Compare the location of the peak with μ.
   (c) $f(x) = \cos x$ on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
   (d) ("Normal distribution") $f(x) = ce^{-\frac{(x-\mu)^2}{2\sigma^2}}$ on the whole line.

5. (Probabilities)
   (a) Let x be distributed uniformly in the interval $[a, b]$. Find the probability that f is more than two standard deviations greater than the mean.
   (b) The heights of Canadian men are approximately normally distributed (see 4(d)) with mean about 175cm and standard deviation about 7cm. Using the method of problem 6(e) below with $n = 10$ estimate the proportion of Canadian men with height between 170cm and 180cm.

Exploration: Numerical Integration

In these problems we will examine some methods for computing integrals numerically. Accordingly, let f be a continuous function defined on the interval $[a,b]$. We will suppose that derivative of f exist as needed and write $M_k = \max\left\{\left|f^{(k)}(x)\right| \mid x \in [a,b]\right\}$. Let $I = \int_a^b f(x)\,\mathrm{d}x$.

6. The midpoint rule.
   (a) Suppose first that $a = -h/2, b = h/2$ for some parameter h (the length of the interval) and consider the auxilliary function
   $$F_{\mathrm{m}}(y) = \int_{-y}^{y} f(x)\,\mathrm{d}x - 2yf(0)\,.$$
   Show that $F_{\mathrm{m}}(0) = F'_{\mathrm{m}}(0) = 0$.
   (*b) Show that $|F''_{\mathrm{m}}(y)| \leq hM_2$ for all $0 \leq y \leq \frac{h}{2}$ and use Taylor's Theorem to conclude that $\left|F_{\mathrm{m}}\left(\frac{h}{2}\right)\right| \leq \frac{M_2 h^3}{8}$.
   SUPP Using the integral form of the remainder in Taylor's Theorem show that $\left|F_{\mathrm{m}}\left(\frac{h}{2}\right)\right| \leq \frac{M_2 h^3}{24}$.
   (d) Suppose that if f is defined on $[a,b]$ and $a \leq x_{i-1} \leq x_i \leq b$. Show that $\left|\int_{x_{i-1}}^{x_i} f(x)\,\mathrm{d}x - hf\left(\frac{x_i + x_{i-1}}{2}\right)\right| \leq \frac{M_2 h^3}{24}$ where $h = x_i - x_{i-1}$.
   (e) Let $x_i = a + \frac{b-a}{n} i$ for $0 \leq i \leq n$ (the uniform partition). Writing $h = \frac{b-a}{n}$, show that
   $$\left|\int_a^b f(x)\,\mathrm{d}x - h\sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right)\right| \leq \frac{M_2(b-a)^3}{24n^2}\,.$$
   RMK The formula is called the "midpoint rule" for evaluation of integrals, since f is evaluated at the middle of every subinterval.
   (f) Approximate $\log 2 = \int_1^2 \frac{\mathrm{d}x}{x}$ to 2 decimal digits using the midpoint rule.
   (g) Approximate $4\int_0^1 \frac{\mathrm{d}x}{1+x^2}$ to 2 decimal digits using the midpoint rule. What is the exact answer?

SUPP (The trapezoid rule)
   (a) On the interval $\left[-\frac{h}{2}, \frac{h}{2}\right]$ use the auxilliary function $F_{\mathrm{t}}(y) = \int_{-y}^{+y} f(x)\,\mathrm{d}x - y\left(f(y) + f(-y)\right)$ to show that
   $$\left|\int_{-h/2}^{+h/2} f(x)\,\mathrm{d}x - h\frac{f(\frac{h}{2}) + f(-\frac{h}{2})}{2}\right| \leq \frac{M_2 h^3}{12}\,.$$
   (b) Conclude that with the notation of 6(d),
   $$\left|\int_a^b f(x)\,\mathrm{d}x - h\left(\frac{f(a)}{2} + \sum_{i=1}^{n-1} f(x_i) + \frac{f(b)}{2}\right)\right| \leq \frac{M_2(b-a)^3}{12n^2}\,.$$
   Hint: keep track of the contribution of each endpoint as you sum over the subintervals $[x_{i-1}, x_i]$.

RMK This is called the "trapezoid rule" since $(b-a)\frac{f(a)+f(b)}{2}$ is the area of the trapezium with vertices $(a,0),(a,f(a)),(b,f(b)),(b,0)$. It is less accurate than the midpoint rule for the same number of function evaluations, but it is simpler and sometimes more convenient.

Supplementary problems

A Let $B_n(R)=\left\{(x_1,\ldots,x_n)\in\mathbb{R}^n \mid \sum_{i=1}^n x_i^2 \leq R^2\right\}$ be the ball of radius R in n-dimensional space. We will calculate its volume.

(a) Show that the 1-dimensional volume of $B_1(R)$ is $2R$.

(b) Suppose that the n-dimensional volume of $B_n(R)$ is c_nR^n. Show that the $(n+1)$-dimensional volume of B_{n+1} is $c_{n+1}R^{n+1}$ where

$$c_{n+1}=2c_n\int_0^{\pi/2}\cos^{n+1}\theta\,\mathrm{d}\theta\,.$$

(c) Show that $c_n=\frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}$.

(d) Find the $(n-1)$-dimensional volume of a sphere of radius R in n-dimesional space. *Hint*: Slice the ball into concentric spheres.