Math 121: Problem set 10 (due 27/3/12)

Practice problems (not for submission!)

Sections 9.2, 9.3: Series.

Sequences and Series

1. (Sanity-check test for series; you will be expected to know this)
   SUPP Let $\sum_{n=1}^{\infty} a_n$ be a convergent series. Show that $\lim_{n\to\infty} a_n = 0$.
   (b) Show that $\sum_{n=1}^{\infty} \frac{(1+(-1)^n)n^2}{(n+1)^2}$ diverges.
2. Summation of series you know how to sum.
   (a) Evaluate $\sum_{k=5}^{\infty} \frac{7}{8^k}$.
   (b) (From past final) Evaluate $\sum_{k=2}^{\infty} 3^{k-1}2^{-2k}$.
   (c) Evaluate $\sum_{k=2}^{\infty} \frac{k+1}{\pi^{2k}}$.
   Hint on the other side.

3. Determine whether the following series converge. If $a_n \sim b_n$ and both are positive then $\sum a_n$ and $\sum b_n$ either both diverge or both converge, so for some problems it is enough to point out the asymptotics. This won't work in all cases!
   (a) $\sum_{n=10}^{\infty} \sqrt{\frac{n^2-5}{n^3+n}}$.
   (b) $\sum_{n=1}^{\infty} \frac{2^{2k-1}}{3^k+k}$
   (c) $\sum_{n=1}^{\infty} \frac{1}{n\log n}$
   (d) $\sum_{n=10}^{\infty} \frac{1}{n\log n(\log\log n)^p}$ (your answer will depend on p).
   (e) $\sum_{n=0}^{\infty} \frac{x^{2n}}{1+2^n}$ (your answer will depend on x).

4. Show that $\sum_{n=1}^{\infty} \frac{n^2\left(1+\frac{1}{n}\right)\cos\left(n^2x\right)}{3^n}$ converges absolutely for all x.
   Hint on the other side.

5. A tail estimate. Let $e = \sum_{n=0}^{\infty} \frac{1}{n!}$.
   (a) Show that $\sum_{n=N+1}^{\infty} \frac{1}{n!} \leq \frac{1}{(N+1)!}\frac{1}{1-\frac{1}{N+2}}$.
   (b) Calculate e to within 0.01.
   (*c) Let $N \geq 1$ be an arbitrary integer. Show that $N!e$ is not an integer, and hence that e is irrational.

6. Review of Taylor expansions
   RMK Recall the Lagrange form of the remainder in Taylor's formula: Let f be $(n+1)$-times differentiable in a neighbourhood of a. Then for any x in the neighbourhood there is ξ between x and a such that
$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}.$$
   (a) Show that for all x, $\lim_{n\to\infty} \left|e^x - \sum_{k=0}^{n} \frac{1}{k!}x^k\right| = 0$, that is that $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$.
   (*b) Show that for $|x| < 1$, $\frac{1}{\sqrt{1-x}} = \sum_{k=0}^{\infty} \binom{2k}{k}\left(\frac{x}{4}\right)^k$.

Hint for 2(c): Change variable (shift the index) and then connect this to $\sum_{k=0}^{\infty} \frac{k}{(\pi^2)^k}$.

Hint for 3*:* Get rid of "red herring" terms using an upper bound before applying d'Alambert's test.

Hint for 5(a): Take out a common factor of $\frac{1}{(N+1)!}$ and compare with a geometric series.

Exam practice: Summing series

WARNING*:* The following problem should not be taken as a form of medical advice.

A. The drug Synthroid (Levothyroxine) has a half-life in the body of about 7 days. Write $\eta = 2^{-1/7}$.

(a) Suppose a patient takes a dose of d milligrams once daily. What quantity remains in her body after n days?

RMK Formally we need d to denote the amout actually absorbed by the patient; we will ignore this issue.

(b) Suppose the patient has been taking the drug at that dose for n consecutive days (including today). What is the current quantity of the drug in the patient?

(c) (Tail estimate) Compare the drug levels in a patient taking the drug for $7-8$ weeks and for "infinitely many weeks". Which has a simpler formula?

DEF Let D be the drug level in the patient in the "infinite-horizon" limit.

(d) Let a_n denote the amount in the patient in day n. Show that $a_{n+1} = \eta a_n + d$, and hence that $a_{n+1} - D = \eta(a_n - D)$.

(e) Suppose a patient with drug level D at day -1 fails to take her medication at day 0, but resumes taking the drug at day 1. At what day will her level return to within 5% of D?

RMK If a patient misses a dose of this drug, it is not recommended to "double up" the next day.

(f) Now consider a drug with a half-life of one day, still taken once daily. Suppose the patient decides to "double up" on day 1. What is the ratio of the level of the drug at day 1 to the "infinite horizon" level?

RMK In the case of short half-life, doubling up can lead to a high dose of the drug, which isn't a good idea either.

B. A promise of one dollar next year is worth η dollars today, where $\eta < 1$ (this is the "discount rate").

(a) How much would you be willing to pay to get d dollars every year for the next 99 years? every year forever? Compare the two numerical values when $\eta = 0.97$.

(b) What is the present value of a promise to pay n^2 dollars in the nth year?

Exam practice: Arithmetic with series

C. Let a_n be the sequence defined by $a_0 = 0$, $a_1 = 1$ and $a_{n+1} = a_n + a_{n-1}$ if $n \geq 1$ (the "Fibonacci numbers")

(a) Show that $0 \leq a_n \leq 2^n$ for all n.

(b) Show that $F(x) = \sum_{n=0}^{\infty} a_n x^n$ converges if $|x| < \frac{1}{2}$.

(c) Show that $F(x) - x = xF(x) + x^2 F(x)$ in the region of convergence.
Hint on the other side.

(d) Let $\varphi, \bar{\varphi}$ be the roots of $x^2 + x - 1 = 0$. Show that $F(x) = \frac{1}{\varphi - \bar{\varphi}} \sum_{n=0}^{\infty} \left(\frac{1}{\varphi^n} - \frac{1}{\bar{\varphi}^n}\right) x^n$ if $|x| < \min\{|\varphi|, |\bar{\varphi}|\}$.

Supplementary problem: e

D. Given a real number x and a positive integer n set $e_n(x) = \left(1+\frac{x}{n}\right)^n$. In this problem we consider the sequence $\{e_n(x)\}_{n=1}^{\infty}$ for x fixed.
   (a) Suppose $n > |x|$.Show that $e_n(x) > 0$; conclude that the sequence is eventually positive.
   (b) Show that if $x < 0$ then $e_n(x) \leq 1$ for $n > |x|$. If $x \geq 0$ show that $e_n(x) \geq 1$ holds for all n.
   (c) Show that $\frac{e_{n+1}(x)}{e_n(x)} \geq 1 + \frac{x^2}{(n+x)(n+1)}$ if $(n+x)(n+1) > |x|^2$. Conclude that the sequence is eventually non-decreasing.
   Hint: Bernoulli's inequality.
   (d) Suppose $x \leq 0$. Show that $\lim_{n\to\infty} e_n(x)$ exists.
   (e) Suppose $x \geq 0$. Show that $e_n(x)e_n(-x) \leq 1$ for all x; use (c) to show that $e_n(x)$ is bounded above, and show that $\lim_{n\to\infty} e_n(x)$ exists in this case too.

E. Let $e(x) = \lim_{n\to\infty} e_n(x)$. In this problem we consider $e(x)$ as a function of x.
   (a) Show that $e(x) \geq 0$ for all x and that $e(0) = 1$. Show that $e(x) \leq 1$ if $x \leq 0$ and $e(x) \geq 1$ if $x \geq 0$.
   DEF Set $e = e(1) = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$.
   (b) Show that $e_2(1) \leq e \leq \frac{1}{e_2(-1)}$ and conclude that $2.25 \leq e \leq 4$.
   (c) Let $x, y \in \mathbb{R}$. Show that $e(x)e(y)e(-x-y) = 1$.
   (d) Deduce that $e(-x) = \frac{1}{e(x)}$ for all x and then that $e(x)e(y) = e(x+y)$ for all x, y.
   (e) Show that $e(n) = e^n$ for all $n \in \mathbb{N}$.
   (f) Show that $e(n) = e^n$ for all $n \in \mathbb{Z}$.
   (g) Show that $e(\frac{p}{q}) = e^{p/q}$ for all $p, q \in \mathbb{Z}$ with $q \neq 0$.
   (**h) Show that $e(x) = e^x$ for all $x \in \mathbb{R}$.

F.
   (a) Show that the coefficient of x^k in $\left(1+\frac{x}{n}\right)^n$ tends to $\frac{1}{k!}$ as n tends to ∞.
   (b) Show that $\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = \sum_{k=0}^{\infty} \frac{x^k}{k!}$.
   Hint: Tail estimates.