Math 121 – Summary of improper integrals

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1 Definitions

- For f defined for $x \geq a$ so that $\int_a^T f(x)\,\mathrm{d}x$ makes sense for all x we set (**IF THE LIMIT EXISTS**)

$$\int_a^\infty f(x)\,\mathrm{d}x = \lim_{T\to\infty} \int_a^T f(x)\,\mathrm{d}x$$

  - Say the integral "converges" if the limit exists, "diverges" if it doesn't.
  - The notation on the LHS is shorthand for the value of the limit.
  - $\int_b^\infty f(x)\,\mathrm{d}x$ converges iff $\int_a^\infty f(x)\,\mathrm{d}x$ converges and in that case $\int_a^\infty f(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_b^\infty f(x)\,\mathrm{d}x$ ("area is additive").
  - Intutition: All that matters is the asymptotic behaviour near infinity.

- For f defined for $a < x \leq b$ we set

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{T\to x} \int_T^b f(x)\,\mathrm{d}x = \lim_{\varepsilon\to 0} \int_{a+\varepsilon}^b f(x)\,\mathrm{d}x\,.$$

  - Again same terminology for "convergence", "divergence".
  - Again if f bounded near b then value of b not important – only behaviour near a is.

- If $\int_a^b f(x)\,\mathrm{d}x$ has several "bad points", break up into sub-intervals on with one bad endpoint each.
  - $\int_0^\infty \frac{e^{-x}}{\sqrt{x|x-2|}}\,\mathrm{d}x = \int_0^1 \frac{e^{-x}}{\sqrt{x|x-2|}}\,\mathrm{d}x + \int_1^2 \frac{e^{-x}}{\sqrt{x|x-2|}}\,\mathrm{d}x + \int_2^3 \frac{e^{-x}}{\sqrt{x|x-2|}}\,\mathrm{d}x + \int_3^\infty \frac{e^{-x}}{\sqrt{x|x-2|}}\,\mathrm{d}x.$

- Limit laws apply, so if the integrals involving f,g on some interval converge so does the integral involving $\alpha f + \beta g$.

2 f positive

- Then $\int_a^T f(x)\,\mathrm{d}x$ is increasing when the interval increases. As $T \to \infty$ it is either bounded (and the limit exists) or unbounded (and the limit is ∞).
  - Key examples:

$$\int_1^\infty \frac{\mathrm{d}x}{\sqrt{x}} = \lim_{T\to\infty} \int_1^T \frac{\mathrm{d}x}{\sqrt{x}} = \lim_{T\to\infty} \left[2\sqrt{x}\right]_1^T = \lim_{T\to\infty} \left(2\sqrt{T} - 2\right) = \infty$$

$$\int_1^\infty \frac{\mathrm{d}x}{x^2} = \lim_{T\to\infty} \int_1^T \frac{\mathrm{d}x}{x^2} = \lim_{T\to\infty} \left[-\frac{1}{x}\right]_1^T = \lim_{T\to\infty} \left(1 - \frac{1}{T}\right) = 1\,.$$

    * In general

$$\int_1^\infty \frac{\mathrm{d}x}{x^p} = \lim_{T\to\infty}\left[\frac{T^{1-p}}{1-p} - \frac{1}{1-p}\right] = \begin{cases} \frac{1}{p-1} & p>1 \\ \infty & p\le 1 \end{cases}$$

  - At a finite interval

$$\int_0^1 \frac{\mathrm{d}x}{x^p} = \lim_{T\to 0}\left[\frac{1}{1-p} - \frac{T^{1-p}}{1-p}\right] = \begin{cases} \frac{1}{1-p} & p<1 \\ \infty & p\ge 1 \end{cases}.$$

- Comparison
  - For f positive, all that matters is the upper bound, so: if $0 \le f(x) \le g(x)$ for all x then
    * If an improper integral for g on some interval converges the same holds for f (smaller area is also finite).
    * If an improper integral for f on some interval diverges the same holds for g (larger area is also infinite).
  - Key situation: suppose for x large f,g are positive and there are constants $0<A<B$ so that $A \le \frac{f(x)}{g(x)} \le B$ for x large. Then $\int_a^\infty f(x)\,\mathrm{d}x$ and $\int_a^\infty g(x)\,\mathrm{d}x$ either both converge or both diverge.
  - Examples for deciding convergence:
    1. $\frac{1}{\sqrt{x^3-5}} \sim_\infty x^{-3/2}$ (asymptotics as $x\to\infty$); since $\int_{10}^\infty \frac{\mathrm{d}x}{x^{3/2}}$ converges so does $\int_{10}^\infty \frac{\mathrm{d}x}{\sqrt{x^3-5}}$.
    2. Decide if $\int_0^1 \frac{e^x\,\mathrm{d}x}{\sqrt{x}}$ converges. Only bad point is at $x=0$; there we have $\frac{e^x}{\sqrt{x}} \sim_0 \frac{1}{\sqrt{x}}$. Since $\int_0^1 \frac{\mathrm{d}x}{\sqrt{x}}$ converges so does $\int_0^1 \frac{e^x}{\sqrt{x}}\,\mathrm{d}x$.
    3. $\int_{1/2}^1 \frac{\mathrm{d}x}{\sin(\pi x)}$. The integrand blows up as $x\to 1$. In what way?
       * Method 1: change variables to $y=1-x$, so we are looking at $\int_{y=1/2}^{y=0} \frac{-\mathrm{d}y}{\sin(\pi y)} = \int_0^{1/2} \frac{\mathrm{d}y}{\sin(\pi y)}$. Now $\sin(\pi y) \sim_0 \pi y$ so $\frac{1}{\sin(\pi y)} \sim_0 \frac{1}{\pi y}$ and the integral diverges.
       * Method 2: (same idea, different presentation) write $\sin(\pi x) = -\sin(\pi x - \pi) = -\sin(\pi(x-1))$. As $x\to 1$, $x-1$ is small so $\sin(\pi(x-1)) \sim_1 \pi(x-1)$. It follows that

$$\frac{1}{\sin(\pi x)} \sim_1 -\frac{1}{x-1} = \frac{1}{1-x}.$$

       Now $\int_{1/2}^1 \frac{dx}{1-x}$ diverges since the integrand blows at at rate $\frac{1}{\text{distance to bad point}}$.

3 Absolute convergence

- Suppose $\int_a^\infty |f(x)|\,\mathrm{d}x$ converges. Then $g(x) = f(x) + |f(x)|$ satisfies $0 \le g(x) \le 2|f(x)|$ so $\int_a^\infty (f(x)+|f(x)|)\,\mathrm{d}x$ converges. Also, $\int_a^\infty (-|f(x)|)\,\mathrm{d}x$ converges. Adding we see that $\int_a^\infty f(x)\,\mathrm{d}x$.
  - If $\int_a^\infty |f(x)|\,\mathrm{d}x$ converges we say $\int_a^\infty f(x)\,\mathrm{d}x$ converges *absolutely*.
  - If $\int_a^\infty f(x)\,\mathrm{d}x$ converges but $\int_a^\infty |f(x)|\,\mathrm{d}x = \infty$ we say $\int_a^\infty f(x)\,\mathrm{d}x$ converges *conditionally*.
- Key examples:
  - $\int_5^\infty \frac{\cos x}{x^2}\,\mathrm{d}x$ converges absolutely since $\left|\frac{\cos x}{x^2}\right| \le \frac{1}{x^2}$.
  - $\int_5^\infty \frac{\cos x}{x}\,\mathrm{d}x$ converges conditionally.