Problem The corner of a paper of size $L \times W$ (L is the long side) is folded so that the corner touches the opposite long side. What is the shortest possible crease?

Solution 1 Let the piece of paper have corners ABCD in cyclic order with $|AB| = |CD| = L$ (long side) and $|BC| = |DA| = W$ (short side). Let $P \in \overline{AB}$, $Q \in \overline{AD}$ be the endpoints of the fold, and let $R \in \overline{CD}$ be the point the corner A lands at after folding. Then the triangles $\triangle PAQ$ and $\triangle PRQ$ are congruent.

Write $x = |AQ|$, $y = |AP|$, and let $z = |PQ| = \sqrt{x^2 + y^2}$ be the length of the fold.

By the congruence above, $|QR| = x$; also $|QD| = W - x$, and since $\triangle QDR$ is right-angled, we see that $|DR| = \sqrt{x^2 - (W-x)^2} = \sqrt{2Wx - W^2}$. By the congruence of the triangles we also have $|PR| = y$. Let ℓ be a line through P parallel to the short sides $\overline{DA}, \overline{BC}$, let let $\overline{CD} \cap \ell = \{P'\}$. Then $\triangle PP'R$ is right-angled, so $y^2 = W^2 + |P'R|^2 = W^2 + \left(y - \sqrt{2Wx - W^2}\right)^2$. We can rewrite this as

$$y^2 = W^2 + y^2 + (2Wx - W^2) - 2y\sqrt{2Wx - W^2}$$

(note that $x \geq \frac{W}{2}$ since $H \leq |AR| \leq |AQ| + |QR| = 2x$ so $\sqrt{2Wx - W^2}^2 = 2Wx - W^2$). Solving for y we find

$$y = \frac{Wx}{\sqrt{2Wx - W^2}}\,.$$

It follows that

$$\begin{aligned} z^2 &= x^2 + \frac{W^2x^2}{2Wx - W^2} \\ &= \frac{2x^3}{2x - W}\,. \end{aligned}$$

To find the location of the minimum we differentiate $Z(x) = z^2(x)$ with respect to x to see:

$$\begin{aligned} Z'(u) &= \frac{6x^2(2x - W) - 2x^3(2)}{(2x - W)^2} = \\ &= \frac{8x^3 - 6Wx^2}{(2x - W)^2} \\ &= \frac{8x^2}{(2x - W)^2}\left(x - \frac{3}{4}W\right)\,. \end{aligned}$$

It follows that the derivative is negative when $x < \frac{3}{4}W$, positive when $x > \frac{3}{4}W$ so there is a minimum when $x = \frac{3}{4}W$.

Note, however, that we must have $y \leq L$. For $x = \frac{3}{4}W$ we have $y = \frac{3}{4}\frac{1}{\sqrt{2\frac{3}{4} - 1}}W = \frac{3}{\sqrt{8}}W$, so the minimum occurs at $x = \frac{3}{4}W$ only if $L \geq \frac{3}{\sqrt{8}}W$ (otherwise the minimum occurs for the longest x possible), that is for the crease such that $P = B$.

Solution 2 Let the piece of paper have corners ABCD in cyclic order with $|AB| = |CD| = L$ (long side) and $|BC| = |DA| = W$ (short side). Let $P \in \overline{AB}$, $Q \in \overline{BC}$ be the endpoints of the fold, and let $R \in \overline{CD}$ be the point the corner B lands on after folding. Then the triangles $\triangle PBQ$ and $\triangle PRQ$ are congruent.

Write $x = |BQ|$, $y = |PB|$, and let $z = |PQ| = \sqrt{x^2 + y^2}$ be the length of the fold.

Let ℓ be a line through P parallel to the short sides $\overline{DA}, \overline{BC}$, let let $\overline{CD} \cap \ell = \{P'\}$. Then $\triangle PP'R$ is right-angled, so $|P'R|^2 = y^2 - W^2$. It follows that $|RC| = y - \sqrt{y^2 - W^2}$.

Next, the triangles $\triangle PP'R$ and $\triangle RDQ$ are similar (they are both right-angled and $\angle PRP' + \angle DRQ = \pi - \angle PRQ = \pi - \frac{\pi}{2} = \frac{\pi}{2}$) so $x = |RQ| = |PR| \frac{|RC|}{|PP'|} = \frac{y}{W}\left(y - \sqrt{y^2 - W^2}\right)$. It follows that

$$\begin{aligned} Z &= z^2 = \frac{y^2}{W^2}\left(y^2 + (y^2 - W^2) - 2y\sqrt{y^2 - W^2}\right) + y^2 \\ &= \frac{2y^4}{W^2} - \frac{2y^3\sqrt{y^2 - W^2}}{W^2} \end{aligned}$$

Dividing by 2 and differentiating,

$$\frac{W^2}{2} Z' = 4y^3 - 3y^2\sqrt{y^2 - W^2} - \frac{y^4}{\sqrt{y^2 - W^2}}$$

So, other than $y = 0$, $Z' = 0$ if

$$4y = \frac{4y^2 - 3W^2}{\sqrt{y^2 - W^2}}$$

so

$$16(y^2 - W^2)y^2 = 16y^4 + 9W^2 - 24y^2W^2$$

so

$$8W^2y^2 = 9W^2$$

so

$$\frac{y}{W} = \frac{3}{\sqrt{8}},$$

if this is in the domain (that is, if $L \geq \frac{3}{\sqrt{8}}W$), at which point

$$x = W\frac{y}{W}\left(\frac{y}{W} - \sqrt{\left(\frac{y}{W}\right)^2 - 1}\right) = \frac{3}{\sqrt{8}}\left(\frac{3}{\sqrt{8}} - \sqrt{\frac{9}{8} - 1}\right)W = \frac{3}{\sqrt{8}} \cdot \frac{2}{\sqrt{8}}W = \frac{3}{4}W.$$

This is the global minimum.

Conclusion: If $L \geq \frac{3}{\sqrt{8}}W$ then the shortest crease occurs when $x = \frac{3}{4}W$. If $L < \frac{3}{\sqrt{8}}W$ the shortest crease occurs when $x = \frac{L}{W}\left(\frac{L}{W} - \sqrt{\left(\frac{L}{W}\right)^2 - 1}\right)W$.

Solution 3 Let the piece of paper have corners ABCD in cyclic order with $|AB| = |CD| = L$ (long side) and $|BC| = |DA| = W$ (short side). Let $P \in \overline{AB}$, $Q \in \overline{AD}$ be the endpoints of the fold, and let $R \in \overline{CD}$ be the point the corner A lands at after folding. Then the triangles $\triangle PAQ$ and $\triangle PRQ$ are congruent.

Write $x = |AQ|$, $y = |AP|$, and let $z = |PQ| = \sqrt{x^2 + y^2}$ be the length of the fold.

Let $r = |DR|$. Then $(W - x)^2 + r^2 = x^2$ so

$$x = \frac{r^2 + W^2}{2W}.$$

Next, from the similarity of the triangles $\triangle PP'R$ and $\triangle RDQ$, $\frac{|PR|}{|PP'|} = \frac{|RQ|}{|RD|}$ so $\frac{y}{W} = \frac{x}{r}$, that is

$$y = \frac{Wx}{r} = \frac{r^2 + W^2}{2r}.$$

Now the constraint $y \le L$ is equivalent to $r^2 + W^2 \le 2rL$, that is $(r - L)^2 \le L^2 - W^2$, or $L - \sqrt{L^2 - W^2} \le r \le L + \sqrt{L^2 - W^2}$, and the constraint $x \le W$ is equivalent to $r \le W$, so $L - \sqrt{L^2 - W^2} \le r \le W$.

It follows that

$$\begin{aligned} 4Z(r) &= 4x^2 + 4y^2 = \left(r^2 + W^2\right)^2 \left(\frac{1}{W^2} + \frac{1}{r^2}\right) \\ &= \frac{(r^2 + W^2)^3}{r^2 W^2}. \end{aligned}$$

We now optimize in r.

$$\begin{aligned} 4Z'(r) &= \frac{6(r^2 + W^2)^2 r}{r^2 W^2} - \frac{2(r^2 + W^2)^3}{r^3 W^2} \\ &= \frac{2(r^2 + W^2)^2}{r W^2} \left(3 - \frac{r^2 + W^2}{r^2}\right) \\ &= \frac{2(r^2 + W^2)^2}{r W^2} \left(2 - \frac{W^2}{r^2}\right). \end{aligned}$$

It follows that if $L - \sqrt{L^2 - W^2} \le \frac{W}{\sqrt{2}}$, that is $L \ge \frac{3}{\sqrt{8}} W$, the minimum occurs when $r = \frac{W}{\sqrt{2}}$, at which point

$$x = \frac{W^2/2 + W^2}{2W} = \frac{3}{4} W.$$

If $W \le L < \frac{3}{\sqrt{8}} W$ then $Z(r)$ is monotone in r and the minimum occurs for the shortest r, that is for $r = L - \sqrt{L^2 - W^2}$, at which point $x = \frac{L}{W} \left(\frac{L}{W} - \sqrt{\left(\frac{L}{W}\right)^2 - 1} \right) W$.