1. Express the following limit as the derivative of a function f at a point a

$$\lim_{h\to 0} \frac{\cos\left(e^7 e^h\right) - \cos(e^7)}{h}$$

Solution: $f(x) = \cos(e^x)$ at $a = 7$ (note that $e^7 e^h = e^{7+h}$).

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2. Differentiate

$$f(x) = x^{2/3}$$

using the definition of the derivative.
You may use the formula

$$(a-b)(a^2+ab+b^2) = a^3 - b^3\,.$$

Solution: For any $x \neq 0$, $h \neq 0$ where $f(x+h)$ is defined,

$$\begin{aligned} \frac{f(x+h)-f(x)}{h} &= \frac{(x+h)^{2/3}-x^{2/3}}{h} \\ &= \frac{1}{h}\frac{\left((x+h)^{2/3}\right)^3 - \left(x^{2/3}\right)^3}{((x+h)^{2/3})^2 + (x+h)^{2/3}x^{2/3} + (x^{2/3})^2} \\ &= \frac{(x+h)^2 - x^2}{h} \cdot \frac{1}{(x+h)^{4/3} + (x+h)^{2/3}x^{2/3} + x^{4/3}}\,. \end{aligned}$$

Now $\lim_{h\to 0} \frac{1}{(x+h)^{4/3}+(x+h)^{2/3}x^{2/3}+x^{4/3}} = \frac{1}{(x+0)^{4/3}+(x+0)^{2/3}x^{2/3}+x^{4/3}} = \frac{1}{3x^{4/3}}$ and $\frac{(x+h)^2-x^2}{h} = \frac{x^2-x^2+2xh+h^2}{h} = 2x+h$ so $\lim_{h\to 0}\frac{(x+h)^2-x^2}{h} = 2x$. From the product rule for limits get

$$\lim_{h\to 0}\frac{(x+h)^{2/3}-x^{2/3}}{h} = \frac{1}{3x^{4/3}}\cdot 2x = \frac{2}{3x^{1/3}} = \frac{2}{3}x^{-1/3}\,.$$

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3. Find the equation of the line that is tangent to the curve $y = e^x$ and passes through the origin.
Note: this is not the same as the line passing through the point $(0,1)$ on the curve.
Solution: Suppose the line meets the curve at the point (a, e^a). Its slope is then $\frac{e^a-0}{a-0} = \frac{e^a}{a}$. The derivative of $y = e^x$ is $y' = e^x$ so being tangent to the curve at (a, e^a), the slope of the line is also e^a. We thus have $\frac{e^a}{a} = e^a$ so $a = 1$. The line then has slope $e^a = e$ so it is the line $y = ex$.