1. Evaluate the following limit:

$$\lim_{x\to 0^+} x^2 \ln x$$

Solution: Since $\lim_{x\to 0} x^2 = 0$ while $\lim_{x\to 0^+} \ln x = -\infty$ this is an indeterminate form, and we may use l'Hôpital's rule. Thus, using $(\ln x)' = \frac{1}{x}$ and $(x^{-2})' = -2x^{-3}$,

$$\lim_{x\to 0^+} \left(x^2 \ln x\right) = \lim_{x\to 0^+} \frac{\ln x}{x^{-2}} = \lim_{x\to 0^+} \frac{1/x}{-2x^{-3}} = \lim_{x\to 0^+} -\frac{x^2}{2} = 0\,.$$

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2. Let

$$f(x) = \frac{1}{x^2 - 9}$$

   (a) What is the domain of f?
   Solution: Since $x^2-9 = (x-3)(x+3)$, the domain is $(-\infty,-3)\cup(-3,3)\cup(3,\infty)$ (equivalently, the real line except for ± 3).

   (b) Find any vertical and horizontal asymptotes.
   Solution: We have $\lim_{x\to-\infty}(x^2-9) = \lim_{x\to+\infty}(x^2-9) = +\infty$ so $\lim_{x\to\pm\infty}\frac{1}{x^2-9} = 0$ and there are horizontal asymptotes on both sides. Also, $\lim_{x\to-3^-}\frac{1}{(x-3)(x+3)} = +\infty$, $\lim_{x\to-3^+}\frac{1}{(x-3)(x+3)} = -\infty$, $\lim_{x\to 3^-}\frac{1}{(x-3)(x+3)} = -\infty$, $\lim_{x\to 3^+}\frac{1}{(x-3)(x+3)} = +\infty$ so there are vertical asymptotes at ± 3.

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3. (Derivatives) Continuing with the same f.

   (a) Where is f increasing? Decreasing? Find all critical numbers and local maxima and minima.
   Solution: By the chain rule, $f'(x) = ((x^2-9)^{-1})' = -(x^2-9)^{-2}(x^2-9)' = \frac{-2x}{(x^2-9)^2}$. This is positive for $x<0$, $x\neq -3$, negative for for $x>0, x\neq 3$ and zero exactly at $x=0$. The function is therefore increasing on $(-\infty,-3)$ and $(-3,0)$, decreasing on $(0,3)$ and $(3,\infty)$ and has a local maximum at $x=0$, which is also the only critical number.

   (b) Where is f concave up? concave down? Find all inflection points. You may use that

$$f''(x) = \frac{6(x^2+3)}{(x^2-9)^3}$$

   Solution: Since $\frac{6(x^2+3)}{(x^2-9)^2} > 0$ where defined, $f''(x) > 0$ when $x^2 > 9$ and $f''(x) < 0$ where $x^2 < 9$. In other words, f is concave up on $(-\infty,-3)$ and $(3,\infty)$ and concave down on $(-3,3)$.