MATH 100 – WORKSHEET 17
EXPONENTIAL GROWTH AND DECAY

1. Exponentials

Growth/decay described by the *differential equation*

$$y' = ky\,,$$

Solution: $y = \boxed{Ce^{kt}}$

(1) A pair of invasive Opossums arrives in BC in 1930. Unchecked, Opossums can triple their population annually. Assume that this in fact happens.
   (a) At what time will there be 1000 Opossums in BC? 10,000 Opossums?
   Solution: Whenver t years pass expect the number of opposums to multiply by 3^t. Thus t years after 1930 expect $2 \cdot 3^t$ opposums. Will have 1000 of them at a time t such that $2 \cdot 3^t = 1000$ so $t = \log_3 500 = \frac{\ln 500}{\ln 3}$ years after 1930. Similarly will reach 10,000 $\frac{\ln 5{,}000}{\ln 3}$ years after 1930.
   (b) Write a differential equation expressing the growth of the Opossum population with time.
   Solution: $y' = (\ln 3)y$.

(2) A radioactive sample decays according to the law

$$\frac{dm}{dt} = -km\,.$$

   (a) Suppose that one-quarter of the sample remains after 10 hours. What is the half-life?
   Solution: Since $\frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2}$ two halvings have occured in 10 hours, so the half-life is 5 hours. Alternatively, suppose $m = Ce^{-kt}$ so that $m(0) = Ce^0 = C$. We know that $m(10) = \frac{1}{4}m(0) = \frac{1}{4}C$ so $Ce^{-10k} = \frac{1}{4}C$ means $-10k = \ln\frac{1}{4}$ and hence $k = \frac{1}{10}\ln 4 = \frac{1}{5}\ln 2$. Now the time $t_{1/2}$ such that $m(t_{1/2}) = \frac{1}{2}C$ will satisfy $Ce^{-\left(\frac{1}{5}\ln 2\right)t_{1/2}} = \frac{1}{2}C$ so $-\frac{1}{5}\ln 2 \cdot t_{1/2} = -\ln 2$. Again we get $t_{1/2} = 5$.
   (b) A 100-gram sample is left unattended for three days. How much of it remains?
   Solution: We have a halving every 5 hours, so $\frac{72}{5}$ halvings over three days. The remaining mass is them $100 \cdot 2^{-\frac{72}{5}}$ grams. Alternatively we could use the formula: $m(72) = 100 \cdot e^{-\frac{1}{5}\ln 2 \cdot 72} = 100 \cdot \left(e^{\ln 2}\right)^{72/5} = 100 \cdot 2^{\frac{72}{5}}$.

(3) Euler found that the tension in a wire wound around a cylinder inreases according to the equation

$$\frac{dT}{d\alpha} = fT$$

where μ is the coefficient of friction and α is the angle around the cylinder.
   (a) When mooring a large ship a rope is wound around a bollard. It is found that when looping the rope once around the bollard, the ratio of tensions at the two ends of the rope is 20. What is the coefficient of friction?
   Solution: The solution to the equation is $T(\alpha) = T(0)e^{f\alpha}$ and we are told that $T(2\pi) = 20T(0)$ so that $e^{2\pi f} = 20$ and $f = \frac{1}{2\pi}\log 20 \approx 0.477$.
   (b) The rope is wound 3.5 times around the bollard. What is the force gain?
   Solution: We gain a factor of 20 every turn, hence a factor of $(20)^{3.5} \approx 35,800$ after 3.5 turns. Akternatively plug in the formula: the gain is $e^{f \cdot 3.5 \cdot 2\pi} = e^{3.5 \cdot \log 20} = 20^{3.5}$.

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2. Newton's Law of Cooling

Fact 1. *When a body of temperature T is placed in an environment of temperature T_{env}, the rate of change of T is* negatively proportional *to the temperature difference $T - T_0$. In other words, there is k such that*

$$T' = -k(T - T_{env})\,.$$

Solving the equation. The key idea is changing variables to the *temperature difference.* Let $y = T - T_{\text{env}}$. Then

$$\frac{dy}{dt} = \frac{dT}{dt} - 0 = -ky$$

so there is C for which

$$y(t) = Ce^{-kt}\,.$$

Solving for T we get:

$$T(t) = T_{\text{env}} + Ce^{-kt}\,.$$

Setting $t = 0$ we find $T(0) = T_{\text{env}} + C$ so $C = T(0) - T_{\text{env}}$ and

$$T(t) = T_{\text{env}} + (T(0) - T_{\text{env}})e^{-kt}\,.$$

Corollary 2. $\lim_{t\to\infty} y(t) = T_0$.

Example (Final Exam, 2010)**.** When an apple is taken from a refrigerator, its temperature is $3°C$. After 30 minutes in a $19°C$ room its temperature is $11°C$.

(1) Write the *differential equation* satisfied by the temperature $T(t)$ of the apple.
Solution: The equation is $T'(t) = k(T(t) - 19)$ and we need to find k. Changing variables to $y = T(t) - 19$ we see that $y' = ky$ so $y(t) = Ce^{kt}$ for some C. Since $y(0) = Ce^0 = C$, we have $C = 3 - 19 = -16$. Now we are given that $-8 = 11 - 19 = y(30) = (-16)e^{30k}$ so $e^{30k} = \frac{1}{2}$. It follows that $k = \frac{1}{30} \ln \frac{1}{2}$ and the equation is

$$\boxed{T'(t) = -\frac{\ln 2}{30}\left(T(t) - 19\right)\,.}$$

(2) Find the temperature of the apple 90 minutes after it is taken from the refrigerator, expressed as an integer number of degrees Celsius.
Solution: We know that $y(t)$ dropped by a factor of 2 after 30 minutes, so over 90 minutes we will see 3 drops of 2, i.e. $y(90) = \frac{1}{8}y(0) = -\frac{16}{8} = -2$. It follows that after 90 minutes the temperature will be $\boxed{T(90) = 19 + y(90) = 17}$ degrees celsius.

(3) Determine the time when the temperature of the apple is $16°C$.
Solution: At that time we have $y(t) = 19 - 16 = -3$ and also $y(t) = -16e^{kt}$. It follows that $e^{kt} = \frac{3}{16}$ so $\boxed{t = \frac{\ln(3/16)}{k} = 30\frac{\ln(16/3)}{\ln 2}}$ minutes.