MATH 100 – WORKSHEET 22
TAYLOR POLYNOMIALS

1. Taylor expansion of e^x

Let $f(x) = e^x$

(1) Find $f(0), f'(0), f^{(2)}(0), \cdots$

Solution: $f^{(k)}(x) = e^x$ for all k, so $\boxed{f^{(k)}(0) = e^0 = 1}$ for all k.

(2) Find a simple polynomial $T_0(x)$ such that $T_0(0) = f(0)$.

Solution: $\boxed{T_0(x) = 1}$ works.

(3) Find a simple polynomial $T_1(x)$ such that $T_1(0) = f(0)$ and $T_1'(0) = f'(0)$.

Solution: We try something like $T_1(x) = 1 + c_1 x$ so we still have $T_1(0) = 1$. Differentiating we see that $(1 + c_1 x)' = c_1$ and since $f'(0) = 1$ we need $c_1 = 1$ so $\boxed{T_1(x) = 1 + x}$.

(4) Find a simple polynomial $T_2(x)$ such that $T_2(0) = f(0)$, $T_2'(0) = f'(0)$ and $T_2^{(2)}(0) = f^{(2)}(0)$.

Solution: Let's try $T_2(x) = 1 + x + c_2 x^2$. Then $T_2(0) = 1$, $T_2' = 1 + c_2 x$ so still $T_2'(x) = 1 = f'(0)$. Finally, $T_2'' = 2c_2$ and we need to match $f''(0) = 1$ so we must take $c_2 = \frac{1}{2}$, i.e. $\boxed{T_2(x) = 1 + x + \dfrac{x^2}{2}}$

(5) Find a simple polynomial $T_3(x)$ such that $T_3^{(k)}(0) = f^{(k)}(0)$ for $0 \le k \le 3$.

Solution: We want to add a term like $c_3 x^3$. Differentiating this three times we get $6c_3$ and this needs to equal 1, so we take $c_3 = \frac{1}{6}$ and $\boxed{T_3(x) = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6}}$.

2. Taylor expansion of $\sqrt{x}$ about $x = 4$

Let $f(x) = \sqrt{x}$

(1) Find a simple polynomial $T_0(x)$ such that $T_0(4) = f(4)$.

Solution: $\boxed{T_0(x) = 2}$.

(2) Find a simple polynomial $T_1(x)$ such that $T_1(4) = f(4)$ and $T_1'(4) = f'(4)$.

Solution: $f'(x) = \frac{1}{2\sqrt{x}}$ so $f'(4) = \frac{1}{4}$. We will try $T_1(x) = 2 + c_1(x-4)$ so that $T_1(4) = 2$ still holds. The derivative of this is c_1 so we want $c_1 = \frac{1}{4}$ and $\boxed{T_1(x) = 2 + \dfrac{1}{4}(x-4)}$.

(3) Find a simple polynomial $T_2(x)$ such that $T_2(4) = f(0)$, $T_2'(4) = f'(4)$ and $T_2^{(2)}(4) = f^{(2)}(4)$.

Solution: $f''(x) = -\frac{1}{4x^{3/2}}$, $f''(4) = -\frac{1}{32}$. We'll take $T_2(x) = 2 + \frac{1}{4}(x-4) + c_2(x-4)^2$. The second derivative is then $2c_2$ so we need $c_2 = -\frac{1}{64}$ and $\boxed{T_0(x) = 2 + \dfrac{1}{4}(x-4) - \dfrac{1}{64}(x-4)^2}$.

(4) Find a simple polynomial $T_3(x)$ such that $T_3^{(k)}(4) = f^{(k)}(4)$ for $0 \le k \le 3$.

Solution: $f^{(3)}(x) = \frac{3}{8}x^{-5/2}$ so $f^{(3)}(4) = \frac{3}{256}$. We need to add a term $c_3(x-4)^3$ whose third derivative is $6c_3$, so we need to take $c_3 = \frac{1}{6}\cdot\frac{3}{256} = \frac{1}{512}$ so $\boxed{T_3(x) = 2 + \dfrac{1}{4}(x-4) - \dfrac{1}{64}(x-4)^2 + \dfrac{1}{512}(x-4)^3}$.

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3. General formula

The nth order Taylor expansion of $f(x)$ about $x = a$ is the polynomial
$$T_n(x) = c_0 + c_1(x-a) + \cdots + c_n(x-a)^n$$
where $c_k = \frac{f^{(k)}(a)}{k!}$.

(1) Find the 4th order expansion of $\frac{1}{1-x}$.

Solution: The first 4 derivatives and corresponding values are

$f^{(0)}(x) = (1-x)^{-1}$	$f^{(1)}(x) = (1-x)^{-2}$	$f^{(2}$
$f^{(0)}(0) = 1$	$f^{(1)}(0) = 1$	

so the expansion is

$$\begin{aligned} \frac{1}{1-x} &\approx 1 + \frac{1}{1}x + \frac{2}{2!}x^2 + \frac{6}{3!}x^3 + \frac{24}{4!}x^4 \\ &= 1 + x + x^2 + x^3 + x^4 . \end{aligned}$$

(2) Find the nth order expansion of $\sin x$.
Solution: See course notes on Taylor Series, pages 4-5.

4. New from old

(1) Find the 3rd order Taylor expansion of $\sqrt{4+x}$ about $x = 0$.
Solution: We know that $\sqrt{u} \approx 2 + \frac{1}{4}(u-4) - \frac{1}{64}(u-4)^2 + \frac{1}{512}(u-4)^3$ for u close to 4. Plugging in $u = 4 + x$ we get $\boxed{\sqrt{4+x} \approx 2 + \frac{1}{4}x - \frac{1}{64}x^2 + \frac{1}{512}x^3}$.

(2) Find the 3rd order Taylor expansion of $\sqrt{4+x} + \frac{1}{1-x}$ about $x = 0$.
Solution: We add the answers to two previous problems to get

$$\begin{aligned} \sqrt{4+x} + \frac{1}{1-x} &\approx \left(2 + \frac{1}{4}x - \frac{1}{64}x^2 + \frac{1}{512}x^3\right) + \left(1 + x + x^2 + x^3\right) \\ &= \boxed{3 + 1\frac{1}{4}x + \frac{63}{64}x^2 + 1\frac{1}{512}x^3}. \end{aligned}$$

(3) Find the 8th order Taylor expansion of $e^{x^2} + \sin(5x)$
Solution: We know that $e^u \approx 1 + u + \frac{u^2}{2} + \frac{u^3}{6} + \frac{u^4}{24}$ where further terms will have higher powers of u. Plugging in $u = x^2$ we get $\boxed{e^{x^2} \approx 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \frac{x^8}{24}}$ where further terms have heigher powers of x. We also know that $\sin u \approx u - \frac{u^3}{3!} + \frac{u^5}{5!} - \frac{u^7}{7!}$ up to 8th order. Plugging in $u = 5x$ we get $\boxed{\sin 5x \approx 5x - \frac{125}{3!}x^3 + \frac{5^5}{5!}x^5 - \frac{5^7}{7!}x^7}$. Adding the two we get

$$\boxed{e^{x^2} + \sin(5x) \approx 1 + 5x + x^2 - \frac{125}{6}x^3 + \frac{1}{2}x^4 + \frac{625}{24}x^5 + \frac{1}{6}x^6 - \frac{5^7}{7!}x^7 + \frac{1}{24}x^8}.$$

(4) Find the 3rd order Taylor expansion of $e^{\sin x} \cdot \cos(\sqrt{x})$.
Solution: To 3rd order we have $\sin x \approx x - \frac{x^3}{6}$, and $e^u \approx 1 + u + \frac{u^2}{2} + \frac{u^3}{6}$. Plugging in $u \approx x - \frac{x^3}{6}$ we get to third order:

$$e^{\sin x} \approx 1 + (x - \frac{x^3}{6}) + \frac{1}{2}(x - \frac{x^3}{6})^2 + \frac{1}{6}(x - \frac{x^3}{6})^3$$

Opening the parantheses any product of terms involving an x^3 times anything will exceed 3rd order so we can drop all terms except:

$$\begin{aligned} e^{\sin x} &\approx 1 + x - \frac{x^3}{6} + \frac{1}{2}x^2 + \frac{1}{6}x^3 \\ &= \boxed{1 + x + \frac{1}{2}x^2}. \end{aligned}$$

Similarly we know that $\cos(u) \approx 1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \frac{u^6}{6!}$ plus terms of order 8 or more. Plugging in $u = \sqrt{x}$ we get

$$\cos\sqrt{x} \approx 1 - \frac{x}{2} + \frac{x^2}{24} - \frac{x^3}{720}$$

plus terms of order 4 or more, so we found the 3rd order expansion. We now multiply:

$$\begin{aligned} e^{\sin x} \cdot \cos(\sqrt{x}) &\approx \left(1 + x + \frac{1}{2}x^2\right)\left(1 - \frac{x}{2} + \frac{x^2}{24} - \frac{x^3}{720}\right) \\ &\approx 1 + \left(-\frac{1}{2} + 1\right)x + \left(\frac{1}{24} - \frac{1}{2} + \frac{1}{2}\right)x^2 + \left(-\frac{1}{720} + \frac{1}{24} - \frac{1}{4}\right)x^3 \\ &= \boxed{1 - \frac{1}{2}x + \frac{1}{24}x^2 - \frac{151}{720}x^3} \end{aligned}$$

where we dropped terms of order greater than 3 in the product.