MATH 100 – WORKSHEET 26
MINIMA AND MAXIMA, MVT

1. More Minima and Maxima

(1) Find the critical numbers of $f(x) = \begin{cases} x^3 - 6x^2 + 3x & x \leq 3 \\ \sin(2\pi x) - 18 & x \geq 3 \end{cases}$.

Solution: $f'(x) = \begin{cases} 3x^2 - 12x + 3 & x < 3 \\ 2\pi\cos(2\pi x) & x > 3 \end{cases}$. Now $3x^2 - 12x + 3 = 3(x^2 - 4x + 1)$ so possible critical points at $\frac{4\pm\sqrt{16-4}}{2} = 2 \pm \sqrt{3}$; but this agrees with f only when $x < 3$ so get a critical point at $\boxed{x = 2 - \sqrt{3}}$. $2\pi\cos(2\pi x) = 0$ iff $2\pi x = \frac{\pi}{2} + \pi k$, $k \in \mathbb{Z}$ so also critical numbers at $\boxed{x = \frac{1}{4} + \frac{k}{2},\ k \in \mathbb{Z}_{\geq 6}}$. $\boxed{x = 3}$ is also a critical number (not differentiable since left/right derivatives don't agree).

(2) Find the absolute minimum and maximum of $g(x) = xe^{-x^2/8}$ on

(a) $[-1, 4]$

Solution: $g'(x) = e^{-x^2/8} - xe^{-x^2/8}\left(-\frac{2x}{8}\right) = \left(1 - \frac{x^2}{4}\right)e^{-x^2/8}$ so critical numbers at $x = \pm 2$. Only $x = 2$ inside interval. We now evaluate f: $f(-1) = -e^{-1/8}$, $f(2) = 2e^{-1/2}$, $f(4) = 4e^{-2}$. f is differentiable so absolute minimum and maximum must occur at endpoints or critical points. Clearly $f(-1)$ is smallest (it's negative) so the absolute minimum is $-e^{-1/8}$. Between the other two, $e > 2$ so $\frac{4}{e^2} < \frac{4}{2^2} = 1$ while $e < 4$ so $\frac{2}{\sqrt{3}} > \frac{2}{\sqrt{4}} = 1$ so $f(2) = 2e^{-1/2}$ is larger and this is the absolute maximum.

(b) $[0, \infty)$

Solution: Only critical point at $x = 2$, $f(2) = 2e^{-1/2}$. Also $f(0) = 0$ and $\lim_{x\to\infty} f(x) = 0$ so indeed $f(2)$ is maximum; 0 is minimum, attained at $x = 0$ (wrong: "also at $x = \infty$" since ∞ is not a number).

(3) Show that the function $3x^3 + 2x - 1 + \sin x$ has no local maxima or minima.

Solution: The derivative is $9x^2 + 2 + \cos x = 9x^2 + 1 + (1 + \cos x) \geq 1 > 0$.

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2. The Mean Value Theorem

Theorem. *Let f be defined differentiable on $[a,b]$. Then there is $a < c < b$ such that $\frac{f(b)-f(a)}{b-a} = f'(c)$. Equivalently, for any x there is c between a, x so that $f(x) = f(a) + f'(c)(x-a)$.*

(1) Let $f(x) = e^x$ on the interval $[0,1]$. Find all values of c so that $f'(c) = \frac{f(1)-f(0)}{1-0}$.
   Solution: Need c so that $e^c = \frac{e-1}{1-0} = e-1$ so $c = \ln(e-1)$ is the only value.
(2) Let $f(x) = |x|$ on the interval $[-1,2]$. Find all values of c so that $f'(c) = \frac{f(2)-f(-1)}{2-(-1)}$.
   Solution: Need c so that $\text{sgn}(c) = \frac{2-1}{2-(-1)} = \frac{1}{3}$ so no value, but f is not differentiable on the whole interval.
(3) Suppose that $f'(x) > 0$ for all x. Show that $f(b) > f(a)$ for all $b > a$. (Hint: consider the sign of $\frac{f(b)-f(a)}{b-a}$).
   Solution: Given a, b there is c such that $\frac{f(b)-f(a)}{b-a} = f'(c) > 0$. Multiply by $b-a > 0$.
(4) Show that $f(x) = 3x^3 + 2x - 1 + \sin x$ has exactly one real zero.
   Solution: Zero exists by IVT: this function is continuous, $f(100) > 0$, $f(-100) < 0$. Two zeroes would contradict monotonicity (checked that $f' > 0$ earlier). Alternative: if $f(a) = f(b) = 0$ then $\frac{f(b)-f(a)}{b-a} = 0$ but $f'(c) \neq 0$ for all c.

Corollary (Monotone function test)**.** *Let f be a function such that f' exists and is continuous on $[a,b]$. Suppose that $f'(x) \neq 0$ for $a < x < b$. Then f has an inverse function on this interval.*