Math 223: Problem Set 5 (due 10/10/12)

Practice problems (recommended, but do not submit)

Calculations with matrices

1. Let $A = \begin{pmatrix} -2 & 3 \\ 5 & -7 \end{pmatrix}$, $B = \begin{pmatrix} 4 & 1 & 0 \\ 0 & -2 & 9 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 2 & 2 \end{pmatrix}$, $D = \begin{pmatrix} 7 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 5 \end{pmatrix}$. Calculate all possible products among pairs of A,B,C,D (don't forget that $A^2 = AA$ is also such a product and that XY, YX are different products if both make sense).

PRAC The $n \times n$ *identity matrix* is the matrix $I_n \in M_n(\mathbb{R})$ with entries: $(I_n)_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$. Show that $I_n \underline{v} = \underline{v}$ for all $\underline{v} \in \mathbb{R}^n$.

2. Let $A \in M_{m,n}(\mathbb{R})$. Show that $AI_n = I_m A = A$. (Hint)

PRAC

(a) Let $A \in M_{n,m}(\mathbb{R})$, $B \in M_{m,p}(\mathbb{R})$. Show that the jth column of AB is given by the product $A\underline{v}$ where $\underline{v}$ is the jth column of B.

(b) Let $A \in M_{n,m}(\mathbb{R})$, $B \in M_{m,p}(\mathbb{R})$. Show that the jth column of AB is a linear combination of all the columns of A with the coefficients being the jth column of B.

3. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R})$ and suppose that $ad - bc \neq 0$.

(a) Find a matrix $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$ such that $AB = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Show that $BA = I_2$ as well.

(*b) ("Uniqueness of the inverse") Suppose that $AC = I_2$. Show that $C = B$.

*4. Find a matrix $N \in M_2(\mathbb{R})$ such that $N^2 = 0$ but $N \neq 0$.

5. ("Group homomorphisms")

(a) Let R_α be the matrix $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$ ("rotation in the plane by angle α"). Show that $R_\alpha R_\beta = R_{\alpha+\beta}$.

(b) Let $n(x)$ be the matrix $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ ("shear in the plane by x"). Show that $n(x)n(y) = n(x+y)$.

An application to graph theory

*6. Let V be a vector space. A linear map $T\colon V \to V$ is said to be *bipartite* if there are subspaces $W_1, W_2 \subset V$ such that $V = W_1 \oplus W_2$ (internal direct sum). and such that $T(W_1) \subset W_2$ and $T(W_2) \subset W_1$. Let T be bipartite with respect to the decomposition $V = W_1 \oplus W_2$. Show that $\dim \operatorname{Ker} T \geq |\dim W_1 - \dim W_2|$.

Hint for 2: interpret the compositions as linear maps, and use the practice problem.
Hint for 3a: use the practice problem and a previous problem set.

Supplementary problems

A. Show by hand that for any three matrices A,B,C with compatible dimensions, $(AB)C = A(BC)$.

B. (Every vector space is $\mathbb{R}^n$) Let V be a vector space with basis $B = \{\underline{v}_i\}_{i\in I}$ (I may be infinite).

   (a) Let $\Phi\colon \mathbb{R}^{\oplus I} \to V$ be the map $\Phi(f) = \sum_{i\in I} f_i \underline{v}_i = \sum_{f_i\neq 0} f_i \underline{v}_i$ [we admit infinite sums if only finitely many summands are non zero]. Show that Φ is a an isomorphism of vector spaces.

   RMK The inverse map $\Psi\colon V \to \mathbb{R}^{\oplus I}$ is called the *coordinate map* (in the ordered basis B)

   (b) Construct an isomorphism $V^* \to \mathbb{R}^I$.

   (c) Let W be another space with basis $C = \left\{\underline{w}_j\right\}_{j\in J}$. Construct an injective linear map $\mathrm{Hom}(V,W) \to M_{I\times J}(\mathbb{R}) = \mathbb{R}^{I\times J}$ and show that its image is the set of matrices having at most finitely many non-zero entries in each column.