Math 223: Problem Set 12 (due 30/11/2012)

Practice problems

Section 6.1

Calculation

1. Let $S = \left\{ \begin{pmatrix} i \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ i+1 \\ 1-2i \end{pmatrix}, \begin{pmatrix} 0 \\ 5+2i \\ 1+2i \end{pmatrix} \right\} \subset \mathbb{C}^3$.
   (a) Calculate the 9 pairwise inner prtoducts of the vectors.
   (b) Calculate the norms of the three vectors (recall that $\|\underline{x}\| = \sqrt{\langle \underline{x}, \underline{x} \rangle}$).

2. Let $S = \left\{ \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\} \subset \mathbb{R}^3$.
   (a) Verify that this is an orthonormal basis of $\mathbb{R}^3$.
   (b) Find the coordinates of the vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix}$ in this basis.

3. Find an orthonormal basis for the subspace $W^\perp \subset \mathbb{R}^4$ if $W = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \right\}$.

4. Using the standard (L^2) inner product on $C(-1,1)$ apply the Gram–Schmidt procedure to the following independent sequences:
   (a) $\left\{1, x, x^2\right\}$ (in that order)
   RMK Applying the Gram–Schmidt procedure to the full sequence $\{x^n\}_{n=0}^{\infty}$ yields the sequence of *Legendre polynomials* $P_n(x)$ (with a non-standard normalizatin).
   (b) $\left\{x^2, x, 1\right\}$ (in that order)

PRAC In each case apply the Gram–Schmidt procedure to the first few members of the sequence $\left\{1, x, x^2, \cdots\right\}$ with respect to the given inner product on $\mathbb{R}[x]$.
(a) (Hermit polynomials) $\langle f, g \rangle = \int_{-\infty}^{+\infty} f(x) g(x) e^{-x^2}\, \mathrm{d}x$.
(b) (Laguerre polynomials) $\langle f, g \rangle = \int_0^{\infty} f(x) g(x) e^{-x}\, \mathrm{d}x$.

Cauchy–Schwarz

SUPP Use induction on n to establish *Lagrange's identity:* for all $\underline{a}, \underline{b} \in \mathbb{R}^n$:

$$\|\underline{a}\|^2 \|\underline{b}\|^2 - (\langle \underline{a}, \underline{b} \rangle)^2 = \left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right) - \left(\sum_{i=1}^n a_i b_i\right)^2 = \sum_{1 \le i < j \le n} \left(a_i b_j - a_j b_i\right)^2$$

(note that the Cauchy–Schwarz inequality for $\mathbb{R}^n$ follows immediately)

5.
   (a) Let $\{x_i\}_{i=1}^n \subset \mathbb{R}$ be n real numbers. Applying the CS inequality to the vectors $(x_1, \ldots, x_n)$ and $(1, \ldots, 1)$, show that $\left(\frac{1}{n} \sum_{i=1}^n x_i\right)^2 \le \frac{1}{n} \sum_{i=1}^n x_i^2$.

RMK The quantities $\frac{1}{n}\sum_{i=1}^{n}x_i$, $\sqrt{\frac{1}{n}\sum_{i=1}^{n}x_i^2-\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)^2}$ are called respectively the *expectation* and *standard deviation* of the random variable that takes the values x_i with equal probability $\frac{1}{n}$.

(**b) Let $\{x_i\}_{i=1}^{n}\subset\mathbb{R}$ be positive. The *Arithmetic Mean* of these numbers is the number AM $=\frac{1}{n}\sum_{i=1}^{n}x_i$. The *Harmonic Mean* is the number $\frac{1}{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{x_i}}=\frac{n}{\sum_{i=1}^{n}\frac{1}{x_i}}$. Show the *inequalithy of the means* HM$\leq$AM (with equality iff all the x_i are equal) by applying the CS inequality to suitable vectors.

Diagonalization

PRAC Check that the eigenvectors of the matrix $\begin{pmatrix}5&4&2\\4&5&2\\2&2&2\end{pmatrix}$ from PS10 are orthogonal.

6. Let $A\in M_n(\mathbb{C})$ be diagonable. Show that there exists $B\in M_n(\mathbb{C})$ such that $B^2=A$.

Orthogonality

7. Let V be an inner product space, $W\subset V$ a subset.
   (a) Show that $\underline{w}^{\perp}=\{\underline{v}\in V\mid\langle\underline{w},\underline{v}\rangle=0\}$ is a subset of V for any $\underline{w}\in V$ (Hint: is this the kernel of something?)
   (b) Show that $W^{\perp}=\{\underline{v}\in V\mid\langle\underline{w},\underline{v}\rangle=0\text{ for all }\underline{w}\in W\}$ is a subspace of V.
   (c) Show that $W^{\perp}\cap\mathrm{Span}_F W=\{\underline{0}\}$.

Supplementary problem: Fourier series

A In this problem we use the standard inner product on $C(-\pi,\pi)$.
   (a) Show that $\left\{\frac{1}{\sqrt{2\pi}}\right\}\cup\left\{\frac{1}{\sqrt{\pi}}\cos(nx),\frac{1}{\sqrt{\pi}}\sin(nx)\right\}_{n=1}^{\infty}$ is an orthonormal system there.
   (b) Let a_0,a_n,b_n be the coefficient of $f(x)=2\pi|x|-x^2$ with respect to $\frac{1}{\sqrt{2n}}$, $\frac{1}{\sqrt{\pi}}\cos(nx)$, $\frac{1}{\sqrt{\pi}}\sin(nx)$. Find these.
   (c) Show that for any x, 1the series $\frac{1}{\sqrt{2\pi}}a_0+\frac{1}{\sqrt{\pi}}\sum_{n=1}^{\infty}\left(a_n\cos(nx)+b_n\sin(nx)\right)$ is absolutely convergent.

   FACT1 The system above is *complete*, in that the only function orthogonal to the span is the zero function. If we denote the partial sums $\left(S_N f\right)(x)=a_0\frac{1}{\sqrt{2\pi}}+\frac{1}{\sqrt{\pi}}\sum_{n=1}^{N}\left(a_n\cos(nx)+b_n\sin(nx)\right)$, this shows $S_N f\xrightarrow[N\to\infty]{}f$ "on average" in the sense that $\|f-S_N f\|_{L^2(-\pi,\pi)}^2=\int_{-\pi}^{\pi}\left|f(x)-\left(S_N f\right)(x)\right|^2\mathrm{d}x\xrightarrow[N\to\infty]{}0$ (in fact, this holds for any f such that $\int_{-\pi}^{+\pi}|f(x)|^2\mathrm{d}x<\infty$).

   FACT2 For any $x\in(-\pi,\pi)$ if the sequence of real numbers $\{\left(S_N f\right)(x)\}_{N=1}^{\infty}$ converges, and if f is continuous at x, then limit of the sequencee is $f(x)$.

   (d) Conclude that $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$, a discover of Euler's.