Math 538: Problem Set on Adèles

Fix a number field K. For a finite (=non-archimedean) place v of K let K_v be the copmletion, $\mathcal{O}_v$ the valuation ring, $\mathfrak{p}_v$ the maximal ideal, $\varpi_v \in \mathfrak{p}_v \setminus \mathfrak{p}_v^2$ a uniformizer, κ_v the residue field, $q_v = \#\kappa_v$ the size of that field. For an infinite (=archimedean) place write K_v for the completion (this is isomorphic to either $\mathbb{R}$ or $\mathbb{C}$).

We normalize the absolute values by $\|\varpi_v\|_v = q_v^{-1}$ in the finite case, and $\|z\|_v$ being the usual absolute value when $K_v \simeq \mathbb{R}$, the square of the usual absolute value when $K_v \simeq \mathbb{C}$. Note that when $K = \mathbb{Q}$ this agrees with our prevoius normalizations.

1. (The product formula)
   (a) Show that for all $x \in K$ and all rational primes p, $\prod_{v|p} \|x\|_v = \left|N_{\mathbb{Q}}^K x\right|_p$.
   (b) Using the absolute value $\|x+iy\| = x^2+y^2$ on $\mathbb{C}$, show that $\prod_{v|\infty} \|x\|_v = \left|N_{\mathbb{Q}}^K x\right|_\infty$ as well.
   (c) Generalize these formulas to a finite extension L/K of number fields.
   (d) Show that for each $x \in K$ there is a finite subset $S \subset |K|$ (containing all the archimedean places) such that if $v \notin S$ then $\|x\|_v \leq 1$ (euivalently, $x \in \mathcal{O}_v$).. Show that when $x \in K^\times$ we can assume that $\|x\|_v = 1$ for all $v \notin S$.
   (e) Obtain the *productc formula*: for all $x \in K^\times$,
   $$\prod_{v \in |K|} \|x\|_v = 1\,.$$

   RMK (d) means that the image of K under the diagonal embedding $K \hookrightarrow \prod_{v\in|K|} K_v$ lies in the set
   $$\prod_{v\in|K|}{}' K_v = \left\{ \underline{x} \in \prod_v K_v \mid \#\{v \mid x_v \notin \mathcal{O}_v\} < \infty \right\}.$$

2. (The restricted direct product) Let V be an index set, and for each $v \in V$ let X_v be a locally compact topological space. Suppose that we have a finite subset $S_0 \subset M$ and for all $v \notin S_0$ a non-empty compact open subset $K_v \subset X_v$.
   (a) Suppose that the set of $v \in V$ such that X_v is non-compact is infinite. Show that $\prod_{v\in V} X_v$ is not locally compact.
   (b) Let $\prod'_{v\in V} X_v = \{\underline{x} \in \prod_v X_v \mid \#\{v \in V \setminus S_0 \mid x_v \notin K_v\} < \infty\}$. Show that the same set would have been obtained if we had increased S_0 (and forgotten about the K_v associated to the places now included in S_0).
   DEF The set is called the *restricted direct product* of the X_v (restricted with respect to the K_v).
   (c) Endow this set with the topology generated by the basis of sets of the form $U = \prod_v U_v$ where $U_v \subset X_v$ is open for all v, and $U_v = K_v$ for all but finitely many V. Show that the resulting topological space is locally compact.
   DEF This topology is called the restricted direct product topology.
   (e) Suppose that f_v are continuous real-valued functions on X_v so that for all but finitely many v, f_v is constant on K_v with $|f_v(k_v)| \leq 1$ for $k_v \in K_v$. Show that the function $f(\underline{x}) = \prod_v f_v(x_v)$ defines a continuous function on $\prod'_v X_v$.
   (d) Suppose that X_v are topological groups (or topological rings) and that the K_v are compact open subgroups (subrings). Show that the resetricted direct product is a topological group

(ring) under the pointwise operations (that is, the operations coming from the usual direct product).

(f) Develop a notion of *restricted tensor product* based on the observation of (e).

DEFINITION. The *ring of Adèles* of K is the product $\mathbb{A}_K \stackrel{\text{def}}{=} \prod'_{v\in|K|} K_v$, restricted with respect to the compact open subrings $\mathcal{O}_v \subset K_v$.

4. (Adeles)
   (a) Show that the *norm* $\|\underline{x}\| = \prod_v \|x_v\|_v$ is a continous function on $\mathbb{A}_K$ such that $\left\|\underline{x}\cdot\underline{y}\right\| = \|\underline{x}\| \left\|\underline{y}\right\|$.
   (b) Show that the image of K in $\mathbb{A}_K$ via the diagonal embedding is discrete (hint: use the product formula to find an open neighbourhood of $0 \in K$ which is disjoint from $K^\times$).
   FACT $K\backslash\mathbb{A}_K$ is compact.

5. Let $\left\{f_j\right\}_{j=1}^J \subset K[x_1,\ldots,x_n]$ be a set of J polynomials in n variables. For any ring R containg K define a set $V(R) = \left\{\underline{a} \in R^n \mid \forall j : f_j(\underline{a}) = 0\right\}$.
   (a) Suppose that R is a topological ring. Show that $V(R)$ is a closed subset of R^n.
   (b) Show that $V(\mathcal{O}_v)$ is compact and open in $V(K_v)$ for all finite places of K.
   (c) Show that there is a natural homeomorphism $V(\mathbb{A}_K) \simeq \prod'_v V(K_v)$ where the product is restricted with respect to $V(\mathcal{O}_v)$.
   (d) Show that $V(K)$ is discrete in $V(\mathbb{A}_K)$.
   RMK In other words, we can interpret $V(\mathbb{A}_K)$ as n-tuples of Adeles or infinite vectors, each coordinate of which is an n-tuple from K_v..

6. (Idèles) Let $\mathbb{I}_K = \mathrm{GL}_1(\mathbb{A}_K) = \mathbb{A}_K^\times$ be the set of invertible Adèles.
   (a) For a fixed place v show that the map $x \mapsto x^{-1}$ is continuous on $K_v^\times$ in the subset topology coming from the inclusion in K.
   (b) Show that the map $x \mapsto x^{-1}$ is not continuous on $\mathbb{I}_K$ in the subset topology induced from the inclusion $\mathbb{I}_K \subset AK$.
   (c) Show that setwise $\mathbb{I}_K$ is the same as the product of $K_v^\times$, restricted with respect to the subsets $\mathcal{O}_v^\times$ for v finite.
   (d) Identify $\mathbb{I}_K$ with the subset $\left\{(x,y) \in \mathbb{A}_K^2 \mid xy = 1\right\}$. Show that the resulting topology is the same as the restricted direct product topology from (c). We always equip $\mathbb{I}_K$ with this topology.
   (e) Show that maps of multiplication and inversion ($x \mapsto x^{-1}$) are continuous in the Idèle topology, so that $\mathbb{I}_K$ is a locally compact topological group.

7. (Idèles 2)
   (a) Show that the norm map $\|\cdot\|$ of 4(a) is continuous on $\mathbb{I}_K$.
   (b) Show that the image of the diagonal embedding $K^\times \hookrightarrow \prod_v K_v^\times$ lies in $\mathbb{I}_K$ and is discrete there.
   (c) Show that the image lies in fact in the set or *norm-one idèles* $\mathbb{I}_K^1 = \{\underline{x} \in \mathbb{I}_K \mid \|\underline{x}\| = 1\}$.
   FACT (Dirichlet Unit Theorem) $K^\times\backslash\mathbb{I}_K^1$ is compact.